Twisted Prisms of Order Polytopes

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I declare that this Master Thesis was
independently authored by myself, with use of the
referenced sources and support.

Thomas Chappell

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1 Introduction

Two-level polytopes are polytopes for which every facet has a parallel hyperplane (or second level) which contains all other vertices of the polytope. This is a seemingly restrictive property, yet despite this several interesting classes of polytopes are two-level. Among these are the Hanner polytopes, order polytopes and independent set polytopes of perfect graphs.

One natural question to ask about two-level polytopes is; which constructions preserve the two-level property? For example, it is relatively simple to see that the Cartesian product of two two-level polytopes is again two-level. As a special case of this, the prism over a two-level polytope is two-level. An interesting construction that does not, in general, preserve the two-level property is the twisted prism. That said, there are classes of two-level polytopes where the property is preserved by taking the twisted prism. For example, in his 1997 paper A.B. Hansen [6] demonstrated that the twisted prisms over the independent set polytopes of perfect graphs are two-level.

Here we present a similar result for the order polytopes, demonstrating that their twisted prisms, called twisted order polytopes, are two-level. The techniques developed here also result in a complete facet description of the twisted order polytope which highlights a link to a different polytope.

Finally we investigate the possibility of extending a volume preserving transfer map discovered by Richard Stanley to the twisted prisms of the order and chain polytopes.

Fundamentals of Polytope Theory

2

This thesis presents work in the field of discrete geometry, particularly in polytope theory. Although it would be naive to think that a complete understanding of the topic could be developed in such a small space, we present here an overview of the fundamentals of polytope theory and so establish the basic language of the entire document. For a more complete introduction to the field, refer to the seminal texts by Günter Ziegler [14] or Branko Grünbaum [5].

2.1 Polytopes

The main objects of study in this thesis are polytopes. Polytopes are convex sets defined by a finite number of points.

Definition 2.1 (Convexity).
A set $S \subseteq \mathbb{R}^d$ is convex if for any $x, y \in S$ and all $0 \leq \lambda \leq 1$ we have,

$$\lambda x + (1 - \lambda) y \in S.$$

Given any set we can ask for the smallest, or inclusion-minimal, convex set containing it. This set is the intersection of every convex set containing the base set and is referred to as the convex hull. Although this minimality condition completely determines the convex hull, it is unwieldy to use in practice. Instead we can use convex combinations.

Definition 2.2 (Convex Hull).
For any set $X \subseteq \mathbb{R}^d$ the convex hull of X is the set $\mathrm{conv}(X)$ which satisfies,

- $X \subseteq \mathrm{conv}(X)$,
- $\mathrm{conv}(X)$ is convex,

◇ For all convex Y with $X \subseteq Y$ we have $\operatorname{conv}(X) \subseteq Y$.

Definition 2.3 (Convex Combination).
Given a finite set of points $\{s_1, \ldots, s_n\} \subseteq \mathbb{R}^d$ and coefficients $0 \le \lambda_i \le 1$ for $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$,

$$\sum_{i=1}^{n} \lambda_i s_i$$

is called a convex combination of the s_i.

Proposition 2.1 (Convex Hull).
Given a set $S \subseteq \mathbb{R}^d$, the convex hull of S is given by all finite convex combinations of S,

$$\operatorname{conv}(S) = \left\{ \sum_{i=1}^{n} \lambda_i s_i : s_1, \ldots, s_n \in S, 0 \le \lambda_1, \ldots, \lambda_n, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

A polytope is then the convex hull of a finite set of points.

Definition 2.4 (Polytope).
For any finite set of points $S \subseteq \mathbb{R}^d$, the set

$$P = \operatorname{conv}(S)$$

is called a polytope.

The dimension of a polytope is defined as the dimension of its affine span. We often assume that the polytopes we deal with are full dimensional. That is, for a polytope $P \subseteq \mathbb{R}^d$ we have $\dim(P) = d$.

2.2 Representations of Polytopes

There are two common ways to represent polytopes, one using points and another using inequalities.

A polytope described as a convex hull of a finite set of points is called a $\mathcal{V}$-representation. The other way to describe polytopes is as a bounded intersection of a finite number of half spaces.

Definition 2.5 (Polyhedron).
A polyhedron $P \subseteq \mathbb{R}^d$ is a convex set defined by vectors $a_i \in \mathbb{R}^d$ and scalars

$b_i \in \mathbb{R}$, with
$$P = \left\{x \in \mathbb{R}^d : a_1^t x \leq b_1, \ldots, a_n^t x \leq b_n\right\}.$$

Note that polyhedra may in general be unbounded. A polytope given as an $\mathscr{H}$-representation is simply a bounded polyhedron. Following the famous Minkowski-Weyl theorem, every polytope can be given as either a $\mathscr{V}$- or $\mathscr{H}$-representation. That said, it can often be computationally difficult to convert between the two representations.

2.3 Examples

Before proceeding further we should look at a few examples of polytopes.

Trivial Examples

There are very few low dimensional polytopes. The zero-dimensional polytope is just a single vertex and the one-dimensional polytope is a line segment. Three-dimensional polytopes are simply polygons. By convention the empty set is also a polytope.

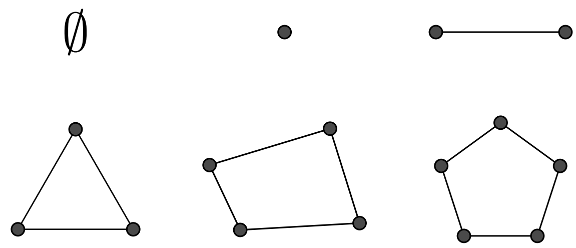

Figure 2.1: Low dimensional polytopes are relatively simple.

Simplex

The simplex is the simplest polytope in each dimension and is the building block of all other polytopes. The minimum number of points required to affinely span a d-dimensional space is $d+1$. Such sets of points form simplices.

Definition 2.6 (Simplex).
The d-dimensional simplex is the convex hull of any $d+1$ affinely independent points.

Typically we use two standard representations of the simplex. One is full-dimensional while the other is more symmetric.

$$\Delta_d = \operatorname{conv}(e_1, \ldots, e_{d+1}) \subseteq \mathbb{R}^{d+1}$$
$$\Delta_d' = \operatorname{conv}(0, e_1, \ldots, e_d) \subseteq \mathbb{R}^d$$

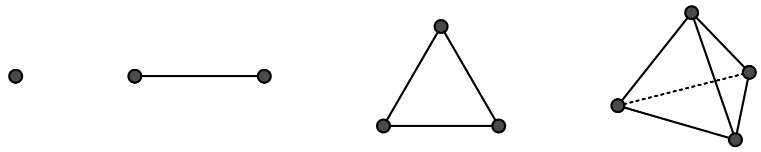

Figure 2.2: Examples of simplicies.

Cube and Cross Polytope

The three-dimensional cube is a familiar object. In discrete geometry we generalise it to all dimensions. Similarly the cross polytope is a generalisation of the octahedron.

Definition 2.7 (Cube)**.**
The cube is the polytope defined by

$$C_d = \{x \in \mathbb{R}^d : 0 \leq x_i \leq 1 \text{ for } i = 1, \ldots, d\} \subseteq \mathbb{R}^d.$$

When dealing with the cube we must always make a decision as to whether it is more sensible to bound between 0 and 1 or -1 and 1. These are commonly referred to as the 0/1- and ±1- cubes. In this thesis we will happily swap between the two affinely-isomorphic representations as the situation demands.

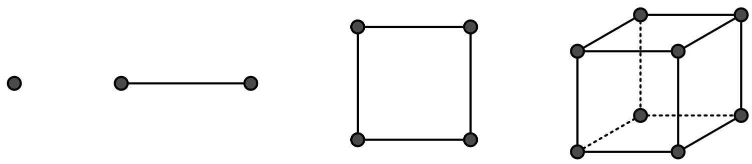

Figure 2.3: Examples of Cubes.

Definition 2.8 (Cross Polytope)**.**
The cross polytope is defined by

$$C_d^* = \operatorname{conv}(\pm e_1, \ldots, \pm e_d) \subseteq \mathbb{R}^d.$$

Note that here we have a perfect of example of polytopes given as $\mathscr{V}$- and $\mathscr{H}$-representations. Although we could give the cube and octahedron using either form, each lends itself naturally to one or the other.

0/1 Polytopes

A 0/1 polytope is a polytope whose vertices are a subset of those of the 0/1 cube. Many of the polytopes we discuss here will be 0/1 polytopes.

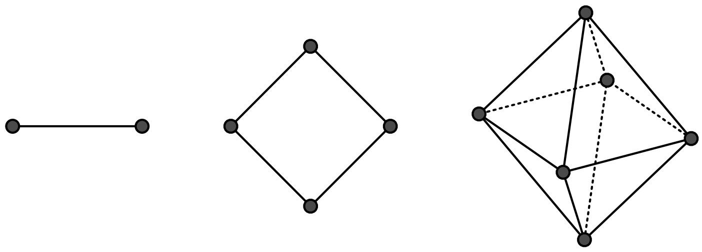

Figure 2.4: Examples of Cross Polytopes.

Definition 2.9 (0/1 Polytope).
A d-dimensional 0/1 polytope is any polytope defined as conv(S) for some $S \subseteq \{0,1\}^d$.

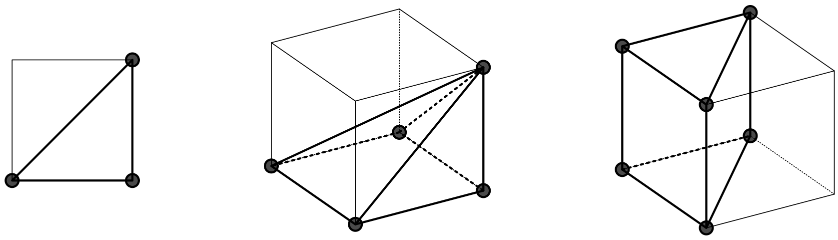

Figure 2.5: A few 0/1 polytopes.

2.4 Faces

The structure of a polytopes boundary presents itself as an inclusion lattice of faces. These faces arise as the intersection of certain hyperplanes with the polytope. In this way we can extract lower dimensional linear features.

Hyperplanes in $\mathbb{R}^d$ can be expressed as equalities $a^t x = b$ for $a \in \mathbb{R}^d$, $a \neq 0$ and $b \in \mathbb{R}$. A valid hyperplane for P is a hyperplane H with

$$a^t x \leq b$$

for all $x \in P$. The intersection of a polytope with a valid hyperplane for that polytope is called a face.

Definition 2.10 (Face).
Given a polytope $P \subseteq \mathbb{R}^d$, a face F of P is given by

$$F = P \cap H$$

for some valid hyperplane H. We write $F \subseteq P$.

Faces of polytopes are themselves polytopes. To see this note that we can write any point in the polytope as a convex combination of a finite set. Since the inequality is valid over the whole polytope, we can write any point that intersects the hyperplane as a convex combination of those members of the aforementioned finite set which intersect the hyperplane.

By convention both the empty set and P itself are considered to be faces of P. Indeed, each of these arises when we choose appropriate supporting hyperplanes in the definition of face. Assuming that P has dimension d, the common nomenclature is that 0-dimensional faces of P are called *vertices*, 1-dimensional faces of P are called *edges* and $(d-1)$-dimensional faces of P are called *facets*.

Given the language of faces, $\mathscr{V}$-representations can be thought of as describing a polytope by giving its vertices and $\mathscr{H}$-representations as giving the facets. Either one provides all the information needed to construct the polytope. Indeed, a $\mathscr{V}$-representation (resp. $\mathscr{H}$-representation) of a polytope which uses the minimal number of points (resp. inequalities) will be exactly the vertices (resp. facets) of the polytope.

Inclusion and the Face Lattice

Every polytope is comprised of its faces and these faces are again polytopes. Even more, faces of faces of a polytope are additionally faces of the original polytope. This pattern of transitivity with face inclusions creates the structure of a partially ordered set on the faces of a polytope.

Definition 2.11 (Face Lattice)**.**
Given a polytope P, the set (P) of faces of P is a poset under the inclusion relation. This poset is called the face lattice of P.

Two polytopes which have isomorphic face lattices are said to be combinatorially equivalent. Essentially this means that the structure of faces and relationships between them are the same. Combinatorial equivalence is one of the main ways to talk about whether polytopes are 'the same'. The other is affine equivalence. Two polytopes are affinely equivalent if there is an affine transformation mapping one to the other. While all affinely equivalent polytopes are combinatorially equivalent the reverse is not true.

Normal Fan

One way of thinking about faces is as the solutions to an optimisation procedure. Consider a polytope $P \subseteq \mathbb{R}^d$. Then, given a direction $a \in \mathbb{R}^d$, the linear function $f(x) = a^t x$ has a finite maximum b on P. This gives a supporting hyperplane $a^t x = b$. The points in the polytope which attain the maximum are the face.

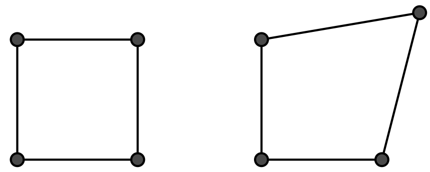

Figure 2.6: Combinatorially equivalent polytopes which are not affinely equivalent.

As a useful notation we write P^a for the face of P obtained by optimising in the direction of a.

Linear optimisation is an incredibly important application of discrete geometry. For our purposes will be most interested in reversing the process. That is, given a specific face $F \subseteq P$ we can ask which vectors v give $P^v = F$.

Definition 2.12 (Normal Cone)**.**
Given a polytope $P \subseteq \mathbb{R}^d$ and a face $F \subseteq P$, the normal cone of F is

$$N_P(F) = \left\{ v \in \mathbb{R}^d : P^v = F \right\}.$$

The normal cone is a geometric cone in the sense that it is closed under positive combinations of its elements. The normal cones of all faces of a polytope form a fan. That is, they partition the affine span of the polytope into a collection of (open) cones with the closures of these cones intersecting in a common face.

Looking at the closed normal cones, we see that the inclusion relation of the faces is reversed in the cones. That is, for faces $F_1, F_2 \subseteq P$ we have

$$F_1 \subseteq F_2 \Leftrightarrow \overline{N(F_2)} \subseteq \overline{N(F_1)}$$

2.5 Common Polytope Notions

Several common notions from polytope theory appear in this document. Each concept is relatively simple but it is worthwhile to be exposed to them here first.

Types of Polytopes

The simplest polytope in any dimension is the simplex. All the facets of the simplex are themselves (lower-dimensional) simplices and each vertex of the d-simplex is contained in exactly d edges. In some sense these two properties characterise the minimality of the simplex. Other polytopes can have one of these properties. Polytopes with the first property, with minimal facets, are called simplicial. Polytopes with the second property, with minimal vertices, are called simple.

Definition 2.13 (Simplicial).
A polytope P is simplicial if every facet of P is a simplex.

Definition 2.14 (Simple).
A (d-dimensional) polytope P is simple if every vertex is incident to d edges.

One useful interpretation of polytopes is as discrete balls. A key property of the sphere is that it is completely symmetrical. Polytopes which are 'sphere-like' and symmetrical around the origin are called centrally symmetric.

Definition 2.15 (Centrally Symmetric).
A polytope P is centrally symmetric if for all $x \in P$ we have $-x \in P$.

Duality

Duality is one of the key concepts in polytope theory. The operation has the effect of interchanging vertices and facets in the polytope.

Definition 2.16 (Combinatorial Dual).
Given a polytope P the combinatorial dual of P is a polytope Q such that there exists an inclusion reversing bijection between the two face lattices.

The geometric definition is a little more intricate. The geometric dual will be a polytope only when the origin is contained in the interior of the polytope. In this case the combinatorics of the geometric dual will match the combinatorial dual.

Definition 2.17 (Dual).
Given a polytope $P \subseteq \mathbb{R}^d$ the dual of P is

$$P^* = \left\{ x \in \mathbb{R}^d : x^t y \leq 1 \ \forall y \in P \right\}.$$

A classic example of duality is the ± 1 cube and cross polytope.

Triangulations and Subdivisions

Triangulations are a very important tool in discrete geometry. An intuitive definition of triangulation is to 'divide a polytope into simplices'. Despite the colloquialism, this definition captures the essence of the idea.

To define triangulation we begin with the more general notion of a subdivision. Here we do not require that all the component parts are simplices.

Definition 2.18 (Subdivision).
Given a polytope P, a subdivision of P is a decomposition of P into full dimensional polytopes P_i where,

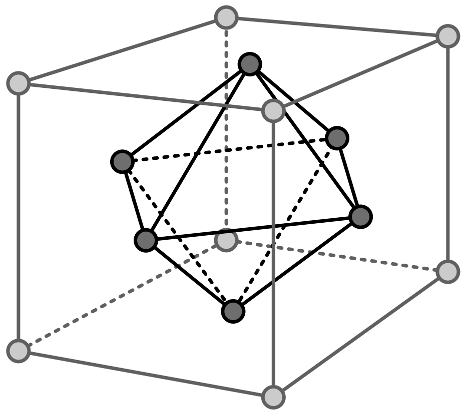

Figure 2.7: The cube and cross-polytopes are duals.

- Every vertex of each P_i is a vertex of P,
- Every P_i is a subset of P,
- Any two of the P_i intersect in a common face,
- The union of the P_i is P.

A triangulation is then just a subdivision where every component is a simplex. The second condition above is, at first glance, somewhat mysterious. To understand the condition, look at the example in Figure 2.8 of a subdivision of the square into triangles that is not a triangulation.

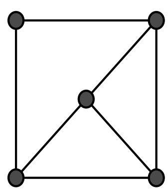

Figure 2.8: A subdivision voiding the second condition of triangulations.

One important technique for constructing triangulations is the pulling triangulation. Given any vertex v of a polytope P, we can subdivide P into pyramids with apex v formed over every facet of P not containing v. We obtain a triangulation by forming similar pyramids over the maximal simplices in a triangulation of each facet not containing v. In this way, we recursively repeat the process by choosing a vertex in each facet to pull from.

Often it is convenient to choose an order for the vertices of P. This then gives a clear way to choose the pulling vertex at each step.

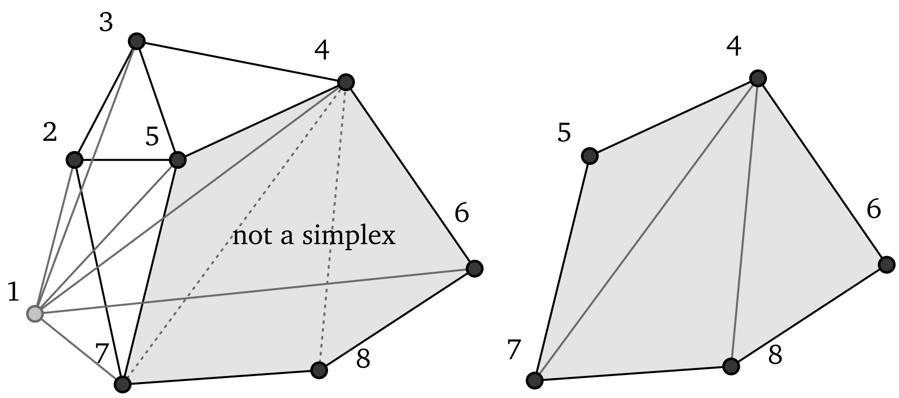

Figure 2.9: Demonstration of a pulling triangulation.

Two-Level Polytopes 3

The main objects of interest in this thesis are so called two-level polytopes. These polytopes are defined by a relatively simple property that involves each of the polytope's facets.

Definition 3.1 (Two-Level Polytope)**.**
Given a polytope P we say that P is two level if for every facet F of P, where F is defined by an inequality $a^t x \leq b$, the inequality takes exactly two values on the vertices of the polytope. That is, there exists a b_1 distinct from b such that for any vertex v of P we have,

$$a^t v = b \text{ or } a^t v = b_1.$$

How should this definition be interpreted? Any facet F of a full-dimensional polytope $P \subseteq \mathbb{R}^n$ defines a normal vector in $\mathbb{R}^n$. The two-level condition then states that in the direction of each of these normal vectors all the vertices of the polytope lie at two distinct heights, or levels. Given this interpretation, the use of the name two-level polytope is hopefully intuitive.

The two-level property seems very restrictive, yet we will see that a surprising number of common classes of polytopes are two-level. In the plane there are only two two-level polytopes, the triangle and square. There are five three-dimensional two-level polytopes. These are the tetrahedron, cube, octahedron, square based pyramid and triangular prism.

For any list of low-dimensional two-level polytopes to make sense we should first clarify what notion of equivalence we are using. Affine transformations preserve parallellity of hyperplanes and as such the affine transformation of a two-level polytope will again be two-level. The same cannot be said for combinatorial equivalence. Consider the slightly distorted square in Figure 3.3. While it is obviously combinatorially equivalent to the regular square, the distorted square is no longer two-level. To avoid such distortions we can always treat two-level polytopes as 0/1 polytopes.

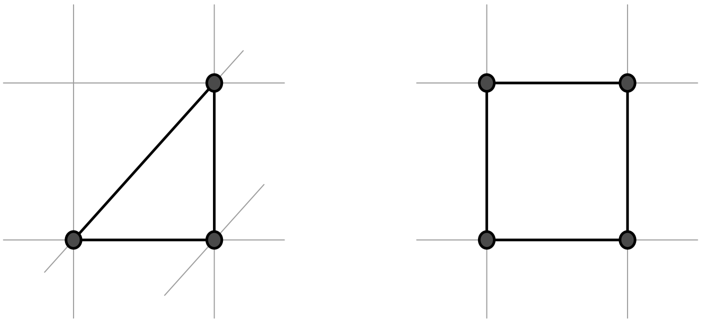

Figure 3.1: The two-dimensional 2-level polytopes.

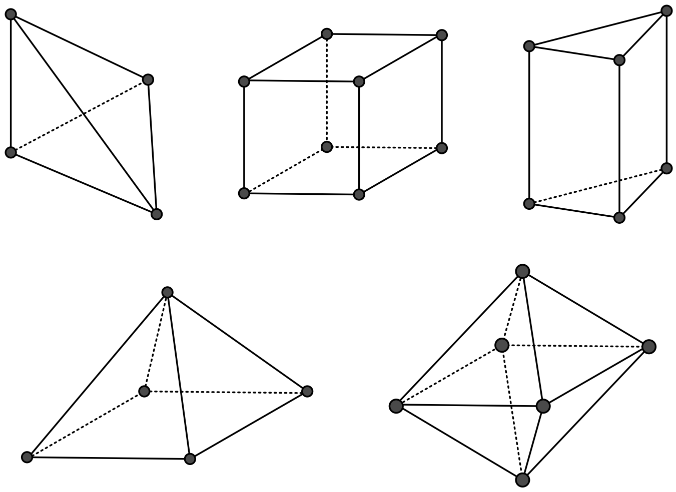

Figure 3.2: The three-dimensional 2-level polytopes.

Proposition 3.1.
Every two-level polytope is affinely equivalent to a 0/1 polytope.

Proof 3.1.
Consider a full-dimensional polytope $P \subseteq \mathbb{R}^d$ and take some d facets of P with linearly-independent facet dimensions. Then from the two-level property we have for each facet two parallel hyperplanes containing all the vertices of P. These hyperplanes form a d-dimensional parallelepiped with the vertices of P contained in the vertices of the parallelepiped. There is then an affine transformation mapping this parallelepiped to the unit cube.

Another useful property of two-level polytopes is that their faces are again two-level. The proof below shows this for facets, but the result extends recursively to all faces.

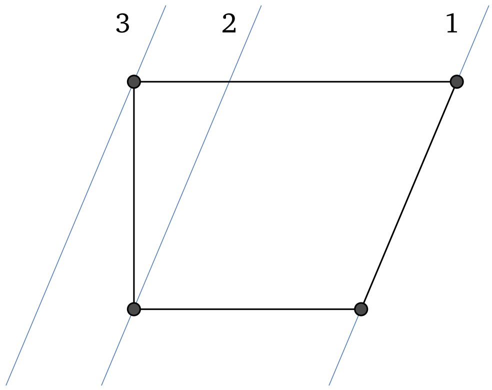

Figure 3.3: The distorted square is not 2-level.

Proposition 3.2.
Every facet of a two-level polytope is two-level.

Proof 3.2.
Take a two-level polytope P and consider any one of its facets, $F \subseteq P$. Suppose that $G \subseteq F$ is a facet of F. Then G is a $(d-2)$-face of P which is defined as the intersection of F with some other facet F_1. Since P is two-level, there is hyperplane H_1 describing the second level of the facet F_1 with all vertices of P contained in either in F_1 or H_1. Then all vertices in F are contained in either G or $H_1 \cap F$. That is, F is two-level.

3.1 Connection to Triangulations

Two-level polytopes have a very interesting connection to triangulations of lattice polytopes. A lattice polytope is any polytope whose vertices lie are integral points. That is, $V(P) \subseteq \mathbb{Z}^d$. Since two-level polytopes are $0/1$ they are also lattice polytopes. The smallest simplex in the integer lattice has volume $\frac{1}{d!}$. We call a triangulation of a lattice polytope unimodular if every simplex in the triangulation has this minimal volume. A polytope in which every pulling triangulation is unimodular is called a compressed polytope.

Definition 3.2 (Compressed Polytope)**.**
A polytope P with vertices on a lattice $L \subseteq \mathbb{R}^d$ is called a compressed polytope if every simplex in every pulling triangulation of P has the minimal volume of any simplex in L.

It turns out that compressed polytopes and two-level polytopes are the same thing! The details of this equivalence come from a paper by Sullivant [13] with further information about compressed polytopes available in a second paper [9].

Proposition 3.3.
Every compressed polytope is two-level and vice-versa.

Proof 3.3.
$(\Rightarrow)$

Suppose that a d-dimensional polytope P is not two-level. So there is some facet F of P defined by $a^t x = b$ and two additional vertices v_1 and v_2 of P with $a^t v_1 = b_1$ and $a^t v_2 = b_2$ for some $b_1 \neq b_2 \neq b$. Consider two pulling triangulations of P which begin with either the vertex v_1 or the vertex v_2. Take a simplex G from the pulling triangulation of F. Then the two pulling triangulations include the cone of v_1 or v_2 over G. However since $a^t v_1 \neq a^t v_2$ the two cones have different volumes.

$$\text{Vol}(v_1 * G) = \frac{a^t v_1}{d}\text{Vol}(G) \neq \frac{a^t v_2}{d}\text{Vol}(G) = \text{Vol}(v_2 * G)$$

Hence at least one of the pulling triangulations of P is not unimodular. That is, the polytope P is not compressed.

$(\Leftarrow)$

We proceed recursively on the dimension of the polytope. Clearly both two-dimensional two-level polytopes are compressed. Suppose P is a d-dimensional two-level polytope and consider any two simplices T_1 and T_2 in a pulling triangulation of P which share a ridge and are pulled from a single vertex v.

If T_1 and T_2 are pulled to v from a single facet F, then by induction $T_1 \cap F$ and $T_2 \cap F$ have the same volume. Therefore T_1 and T_2 have the same volume. If T_1 and T_2 are pulled from different facets there is an affine transformation which maps the two facets to lie on facets of the cube. Again the height between levels is the same and the two facets lie on isomorphic lattices so T_1 and T_2 have the same volume.

Every simplex in the triangulation contains the first pulling vertex and the dual graph of the triangulation is connected. Hence all simplices in the triangulation have the same volume. In the embedding as a 0/1 polytope at least one facet of the polytope will lie on a facet of the cube. Hence all simplices in the triangulation have volume $\frac{1}{d!}$ and the triangulation is unimodular.

3.2 Duality for Two-Level Polytopes

The dual of a two-level polytope is not always two-level. Consider the triangular prism in three dimensions. This is a two-level polytope whose dual is the bipyramid over a triangle. The bipyramid is however not two-level. To see this, observe the particular embedding of the polytope shown in Figure 3.4. That is, with vertices $0, e_1, e_2, e_3, v$. If the bipyramid were two-level then we must have $v = (1,1,1)$. However the facets containing v then take two levels over the remaining vertices.

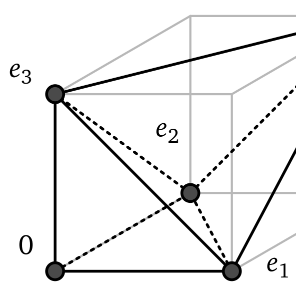

Figure 3.4: The bipyramid over a triangle is not two-level.

For those two-level polytopes which are centrally symmetric, duality does preserve the two-level property.

Proposition 3.4.
The dual of a centrally symmetric two-level polytope is two-level.

Proof 3.4.
Suppose a facet F of a centrally symmetric 2-level polytope P is described by $ax = 1$. Then combining the two properties we know that for all vertices v of P, either $av = 1$ or $av = -1$. Now consider any specific facet F of the dual polytope P^*. This facet corresponds to some vertex v of P and is described by the inequality $v \cdot x = 1$. Similarly, suppose w is any vertex of P^*. Then this w corresponds to a facet of P with the equation $w \cdot x = 1$. Since P is two-level we have $w \cdot v = \pm 1$. This is exactly the equation we need to demonstrate that P^* is two-level.

In [6] A.B. Hansen refers to such centrally symmetric two-level polytopes as weak Hanner polytopes or WHPs. Here they are defined by the condition that for any facet F, the polytope P is given by $P = \mathrm{conv}(F, -F)$. This property is equivalent to being two-level and centrally symmetric.

Hanner polytopes are polytopes which arise as the product, direct sum and dual of smaller Hanner polytopes, beginning with the interval. These polytopes are all centrally symmetric and two-level, and are the extremal cases of several interesting open conjectures.

The 3^d conjecture [7] states that all centrally symmetric d-dimensional polytopes have at least 3^d non-empty faces. This inequality is tight on the Hanner polytopes and it is also conjectured that these are the only polytopes to attain the lower bound. The Mahler conjecture is a similar conjecture about Mahler volumes of centrally symmetric polytopes where again the Hanner polytopes provide the lower bound.

The hope behind studying weak Hanner polytopes is that they may shed some light on these important conjectures given that they share several properties with the Hanner polytopes.

3.3 Twisted Prisms

One technique to create centrally symmetric polytopes is the twisted prism construction [10].

Definition 3.3.
Given a polytope P the twisted prism of P is,

$$\mathscr{T}(P) = \operatorname{conv}(P \times \{1\}, -P \times \{0\}).$$

The analogy to prisms should be clear. When constructing twisted prisms, the choice of height separation between the two copies of P is arbitrary. Typically we use heights of 0 and 1 but in certain situations another choice makes more sense. The important characteristic of a 0/1 polytope is that the vertices are a subset of the vertices of the cube. If we instead use the ± 1 cube and construct our twisted prism with heights of -1 and 1, the twisted prism of a 0/1 polytope will again be a 0/1 polytope. This simple transformation will be important when we consider the twisted prisms of two-level polytopes as all two-level polytopes are 0/1 polytopes.

In general, the twisted prism of a two-level polytope is not two-level. A counterexample, provided by Samuel Fiorini, is given by the following polytope which arises as a subset of the five-dimensional cube,

$$P = \left\{ x \in \mathbb{R}^5 : 0 \leq x_i \leq 1, 1 \leq \sum_{i=1}^{5} x_i \leq 2 \right\}$$

It is clear that this polytope is two-level as the vertices are all integer points and the inequalities given can thus only take two values. However the twisted prism of P is not two level. Assume we have constructed the twisted prism by adding an additional coordinate x_0 taking values 1 and -1 at the copies of P and $-P$ respectively. Consider the inequality

$$x_1 + x_2 + x_3 - x_0 \leq 1.$$

This inequality is facet-defining for the twisted prism. To see this, observe that the inequality is satisfied with equality on exactly six vertices of the twisted prism,

$$(-1,0,0,0,-1,-1), (-1,0,0,0,-1,0), (-1,0,0,0,0,-1),$$
$$(1,0,1,1,0,0), (1,1,0,1,0,0), (1,1,1,0,0,0)$$

These six vertices span a five-dimensional polytope.

Despite the fact that it is facet defining, the above inequality takes more than two values on the vertices of the twisted prism. For example, the inequality takes the values 0, 1 and -1 on the vertices $(1,1,0,0,0,0)$, $(1,1,1,0,0,0)$ and $(1,0,0,0,1,0)$ respectively. Hence the twisted prism is not two-level.

We will be interested in finding situations where the twisted prisms construction does preserve the two-level property. Note that in these circumstances, the constructed polytopes will all be centrally symmetric and hence weak Hanner polytopes.

Independent Set Polytopes 4

Independent set polytopes are a class of polytope related to graphs which have an intriguing relation with the two-level property. Namely, an independent set polytope is two-level if and only if the graph it corresponds to is perfect. These polytopes will provide the first and motivational situation where the two level property is preserved by the twisted prism construction.

4.1 Graph Theory

The notion of a graph is familiar to every mathematician but the choice of precise definition seems to be mostly a matter of taste and not particularly enlightening on its own. For the purposes of this document the following definition will do.

> **Definition 4.1** (Graph).
> A graph G is a pair $G = (V, E)$ where V is a set of vertices and $E \subseteq \binom{V}{2}$ is a set of edges.

Note that the definition forbids loops on single vertices and multiple edges spanning two vertices. Such graphs are often called simple. We call two vertices v, w adjacent if $\{v, w\}$ is an edge.

The independent set polytope arises from the so called independent sets in a graph. A set of vertices is called independent if no two vertices in the set are connected by an edge. The complementary object is a clique, a set of vertices with every possible connection present.

> **Definition 4.2** (Independent Set).
> An independent set (or stable set) in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$, no two of which are adjacent.

> **Definition 4.3** (Clique).
> A clique in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$, every two of which

are adjacent.

As a convenient abuse of notation, we will usually write $S \subseteq G$ to refer both to a subgraph of G and a subset of the vertices.

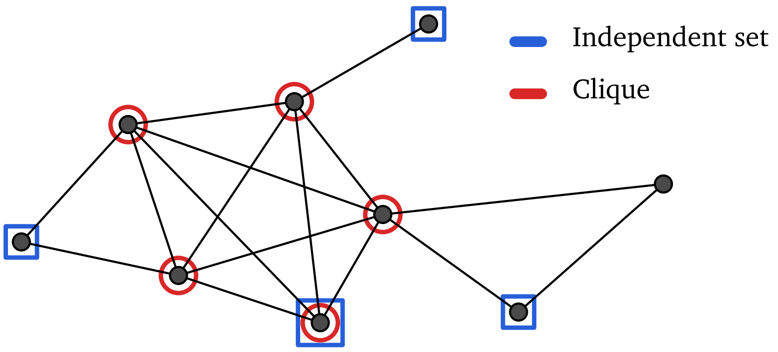

Figure 4.1: An independent set and a clique.

Clearly cliques and independent sets are in some way similar notions. Given a graph we can define its complement by keeping the same vertices and using all the edges not present in the original graph.

Definition 4.4 (Graph Complement)**.**
Given a graph $G = (V, E)$, the graph complement $\overline{G}$ is a graph on the same vertices such that for all vertices $v, w \in V$ we have that $\{v, w\}$ is an edge of $\overline{G}$ if and only if $\{v, w\}$ is not an edge of G.

With this notion of complement we can immediately see that any independent set in G will be a clique in $\overline{G}$ and vice-versa.

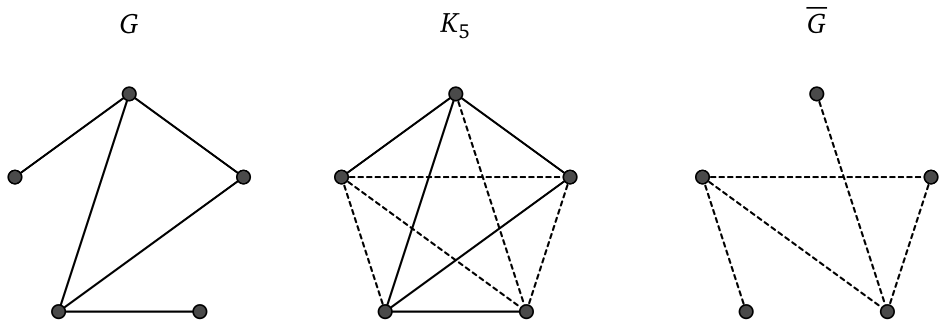

Figure 4.2: The complement of a graph.

Also important for the definition of a perfect graphs is the idea of a vertex colouring.

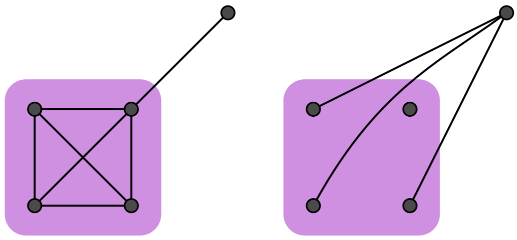

Figure 4.3: Cliques and independent sets reverse in the complement graph.

Definition 4.5 (Vertex Colouring).
A vertex colouring of a graph G is an assignment of colours to each vertex of G such that no two adjacent vertices have the same colour.

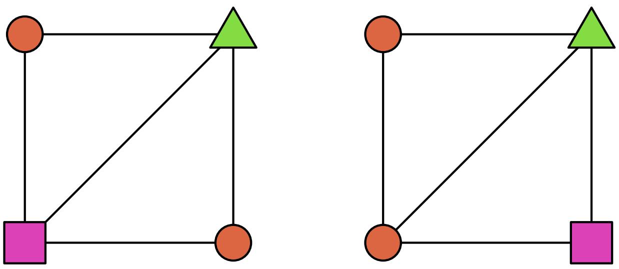

Figure 4.4: Valid and invalid vertex colourings of a graph.

Perfect graphs arise when we attempt to connect the idea of cliques (or equivalently independent sets) with vertex colourings. Why should we even think to do this? In one sense there is a very clear connection between the two. A complete graph on n vertices can only be vertex coloured with n colours. So any graph containing a clique of size n needs at least n colours in a vertex colouring.

Definition 4.6 (Clique and Independence Numbers).
Given a graph G the clique and independence numbers, $\omega(G)$ and $\alpha(G)$, are the sizes of the largest clique and independent set in G respectively.

Definition 4.7 (Chromatic Number).
Given a graph G the chromatic number $\chi(G)$ is the minimum number of colours required to colour the vertices of G such that no two adjacent vertices share a colour.

We know that in every graph the chromatic number is no less than the clique number, $\omega(G) \leq \chi(G)$. In situations where the clique and chromatic numbers of a graph match, the graph is called perfect. Importantly we require that they match not only on the graph itself but on every induced subgraph.

Definition 4.8 (Perfect Graph).
A graph G is called perfect if for every induced subgraph H of G, the chromatic and clique numbers of H are the same.

An important theorem in graph theory is the (weak) perfect graph theorem. This theorem states that a graph is perfect if and only if its complement is. Although a diversion, it is interesting to note that in working with independent set polytopes we will obtain a proof of the weak perfect graph theorem. Indeed it was this theorem, rather than polytopes, that inspired the original proof.

Theorem 4.1 (Perfect Graph Theorem).
The complement of a perfect graph is perfect.

4.2 Independent Set Polytopes

To construct independent set polytopes we relate vertices in the graph to elements of a vector space. This may be done by considering the space of linear functionals on the vertices, $f : V \to \mathbb{R}$, or simply by ordering the vertices and identifying each vertex with a basis vector. We take the latter approach here. That is to say that we relate a graph G with n vertices $v_1, \dots, v_n$ to the n-dimensional vector space $\mathbb{R}^n$ by associating each vertices v_i with the basis vector e_i.

For any subgraph $S \subseteq G$ of a graph G with n vertices, the incidence vector of S is given by

$$x_S = \sum_{v_i \in S} e_i \subseteq \mathbb{R}^n.$$

This incidence vector has coordinates that are 0 or 1 and can be thought of as describing which vertices belong in S. The independent set polytope is constructed using the incidence vectors of all independent sets in a graph.

Definition 4.9 (Independent Set Polytope).
Consider a graph G with n vertices. The independent set polytope associated to G is an n-dimensional polytope given by

$$P_G = \mathrm{conv}\{x_S : S \subseteq G \text{ independent}\}.$$

An example of an independent set polytope is given in Figure 4.5.

Independent set polytopes are full dimensional and always contain a simple vertex. To see this, consider that the empty set and each singleton are always independent sets of a graph. Furthermore, independent set polytopes are 0/1 polytopes as their vertices are 0/1 vectors.

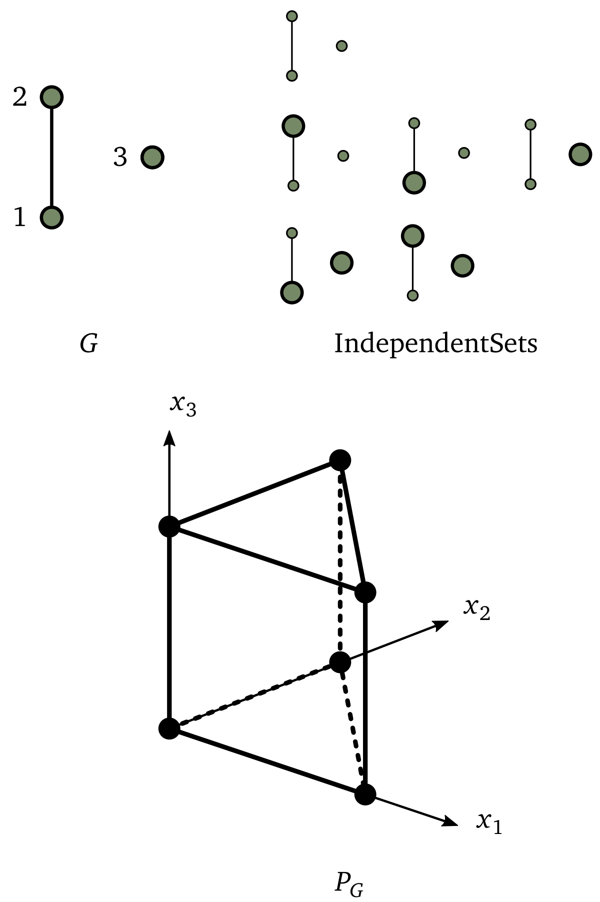

Figure 4.5: Example of an independent set polytope.

4.3 Two-Level Independent Set Polytopes

Given that independent set polytopes are 0/1, one might wonder whether such polytopes are also two-level. The answer to this question was provided by Lovász in [8].

Theorem 4.2 (Lovász)**.**
A graph G is perfect if and only if the independent set polytope P_G is 2-level.

The proof of this theorem depends upon a representation of the independent set polytope with a set of inequalities. Earlier we established a connection between independent sets and cliques in a graph. This connection comes into play here. We will show that for a perfect graph, the vertices of the independent set polytope correspond to independent sets and the facets correspond to cliques.

For any graph G with n vertices we have an $\mathscr{H}$-polytope P'_G defined by,

$$P'_G = \{x \in \mathbb{R}^n : x_i \geq 0, x_K \cdot x \leq 1 \text{ for all cliques } K \subseteq G\}.$$

At this point, we can begin to see a connection between complements of graphs and duals of polytopes. A restricted form of dual called the antiblocking polytope arises when we consider only the positive quadrant.

Definition 4.10 (Antiblocking Polytope).
Given a polytope P the antiblocking polytope $\overline{P}$ is defined by

$$\overline{P} = \{x \in \mathbb{R}^n : x_i \geq 0, x \cdot y \leq 1 \text{ for all } y \in P\}$$

or equivalently

$$\overline{P} = \mathbb{R}^n_+ \cap P^*.$$

With the above notation our clique polytope P'_G can also be written as $\overline{P_{\overline{G}}}$. To prove Lovász' theorem we will show that the independent set polytope P_G and $P'_G = \overline{P_{\overline{G}}}$ coincide exactly when G is perfect. Doing so connects the notions of graph complement and antiblocking polytopes. Namely we will have shown that for perfect graphs,

$$\overline{P_G} = P_{\overline{G}}.$$

Lemma 4.1.
A graph G is perfect if and only if $P_G = P'_G$

This lemma was the original proof of Lovás. It is relatively easy to extend this lemma to incorporate the two-level property.

Corollary 4.1.
Given the above lemma we can prove the main theorem. That is, that a graph G is perfect if and only if the independent set polytope P_G is two-level.

Proof 4.1.
($\Rightarrow$) Suppose that the graph G is perfect. Consider any vertex v and facet F of the independent set polytope P_G. From the lemma, this F is described by $a \cdot x \leq 1$ for some a, the incidence vector of a clique of G. Since v is the incidence vector of an independent set of G and cliques and independent sets can share at most one vertex, the value of $a \cdot v$ can only be 0 or 1. This established that P_G is two-level.

($\Leftarrow$) On the other hand, suppose that P_G is two-level. Consider any facet F of P_G defined by the inequality $a \cdot x \leq 1$. The empty set is independent, so the origin in a vertex of P_G and $a \cdot 0 = 0$. Since P_G is two-level, the function $u \cdot x$ can only take two values on the vertices. Up to scaling we can use 0 and 1. Each singleton in the graph is independent and thus all coordinates of a must be 0 or 1. Finally a cannot have ones in the coordinates corresponding to two non-adjacent vertices as then we would have $a \cdot v = 2$ for some vertex

$v \in P_G$. So a corresponds to a clique in G and hence $P_G = P'_G$.

We now wish to prove the lemma, that is $P_G = P'_G$ if and only if the graph G is perfect. We can immediately see that all graphs satisfy the inclusion

$$P_G \subseteq P'_G.$$

This is because any independent set and clique in a graph can intersect in at most one vertex. So $x_S \cdot x_K \leq 1$ and hence $P_G \subseteq P'_G$.

If $P_G = P'_G$ then the vertices of P'_G are clearly 0/1 points and hence integral. Given the inequalities that define P'_G, the coordinates of a vertex are bounded by 0 and 1. So to say that the polytope P'_G is integral is to say that the coordinates are either 0 or 1. Since a clique and an independent set can only intersect in at most one point, this means that $P_G = P'_G$. Therefore, to say that $P_G = P'_G$ is the same as to say that the vertices of P'_G are integral.

Proposition 4.1.
If G is perfect then $P_G \supseteq P'_G$.

To prove this we will use replicated graphs. Given a graph G and a vertex v, we replicate v by creating a new vertex v' connected to all neighbours of v and to v itself.

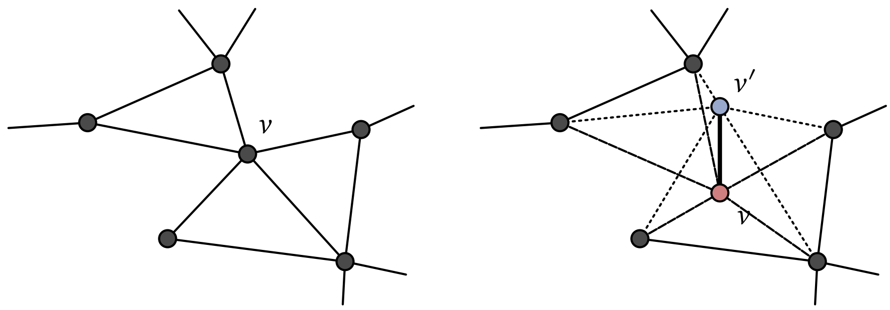

Figure 4.6: A replicated vertex.

An integer weighting of the vertices of a graph is an assignment of a positive integer to each vertex in the graph. This can be thought of as an integral point in the associated vector space. For any positive integer weighting w, we can create a replicated graph G_w by replicating each vertex v_i, multiple times corresponding to the weight w_i.

Lemma 4.2.
For any positive integer weighting w, a graph G is perfect if and only if the replicated graph G_w is.

Proof 4.2.
The original graph G is an induced subgraph of G_w so the reverse direction of the lemma follows directly from the definition of perfect. In the other direction we can consider the individual operations used to create G_w. Note that all induced subgraphs of G_w are either induced subgraphs of G or their replicates. As such it suffices to show that the clique and chromatic numbers of G_w itself match.

Suppose we replicate a vertex v in G.

If we assume that v is in a maximal clique of G, then the replicated vertex is in a clique with one more element. That is, the clique and chromatic numbers both increase by exactly one. Thus $\omega(G_w) = \chi(G_w)$.

Otherwise, suppose v is not in a maximal clique of G. Since G is perfect we have a colouring of G with $\omega = \omega(G)$ colours. Call the coloured vertex sets $A_1, \ldots, A_\omega$ and assume $v \in A_\omega$. Then consider the graph $G - (A_\omega - v)$ which has clique size at most $\omega - 1$ since v was not in a maximal clique. Furthermore it is perfect and so can be coloured in $\omega - 1$ colours. Call these classes $B_1, \ldots, B_{\omega-1}$. Then $B_1, \ldots, B_{\omega-1}, A_\omega$ is a colouring of G_w with one copy of v in A_ω and one in one of the B_i. In this case the chromatic and clique numbers don't change and once again $\omega(G_w) = \chi(G_w)$.

We saw before that $P_G \subseteq P'_G$ and now must show that the converse holds when G is perfect.

Suppose x is a rational point in P'_G and choose some integer N so that $w = Nx$ is integral. Then this w is a weighting on the vertices of G and so we may construct the graph G_w. Note that this G_w is perfect since G is.

Since x is a point in P'_G, we know that for any clique K of G, $x \cdot x_K \leq 1$. For any clique K in G, the replicates of K will be a clique in G_w. Furthermore, any clique K_w in G_w is contained in such a replicated clique from G. Thus the size of the clique K_w is bounded by the number of replicates in the clique K in G which it projects to.

Consider any clique K_w of G_w and the projected clique K in G. We then have

$$\begin{aligned}
|K_w| &\leq \sum_{v_i \in K} w_i \\
&= w \cdot x_K \\
&= N(x \cdot x_K) \\
&\leq N
\end{aligned}$$

So the replicated graph has cliques of size at most N and thus, since it is perfect, can be coloured with N colours. Each of the N sets of vertices $A_1, \ldots, A_N$ in each colour are independent and cover all of G_w. Furthermore since any replicates of a single vertex in G are connected, each A_i can be projected to an independent set S_i in G. This gives the vertices of G_w as a union of copies of independent sets in G and hence w is a sum of N incidence vectors of independent sets of G. Writing

$$\begin{aligned} x &= \frac{w}{N} \\ &= \frac{\sum_{i=1}^{N} x_{s_i}}{N} \end{aligned}$$

then gives x as a convex combination of incidence vectors of independent sets in G. That is, $x \in P_G$.

All together this give $P_G \supseteq P'_G$ when G is a perfect graph and proves proposition 4.1.

Proposition 4.2.
If $P_G = P'_G$ then the complement of G, $\overline{G}$ is perfect.

In the definition of perfect graph we need to check that the clique and chromatic numbers are the same for all induced subgraphs of G. Fortunately we can avoid this here because if $H \subseteq G$ is an induced subgraph and $P_G = P'_G$ then we will also have $P_H = P'_H$. Again we will use integrality of P'_G. Our observation then reduces to the statement that if P'_G is integral, then so is P'_H for any induced subgraph $H \subseteq G$. To see this, consider removing just one vertex v from G. Then $P'_{G-\{v\}}$ is found embedded as a facet of P'_G by setting the coordinate corresponding to v to zero. Since the facet of an integral polytope is clearly also integral, P'_H is integral.

Recall that $\alpha(G)$ is the size of the maximal independent set in G. Since we are attempting to show that the complement of G is perfect, we are interested in the clique number of $\overline{G}$, which is exactly $\alpha(G)$.

We proceed by induction of the number of vertices in G. If G is the trivial graph it is perfect.

For larger G, consider all maximal independent sets of G. The corresponding vertices span a face of P_G defined by the inequality $\sum x_i \leq \alpha(G)$. Since by assumption $P_G = P'_G$ there must be a clique K of G such that this clique defines the face above. The clique K intersects each maximal independent set exactly once, thus for the graph $G \setminus K$ we have $\alpha(G \setminus K) = \alpha(G) - 1$. Taking the complement of $G - K$ gives a graph with clique number $\omega(\overline{G-K}) = \alpha(G) - 1$. By induction there is a colouring of $G - K$ with $\alpha(G) - 1$ colours. The subgraph K is independent in the inverted graph, so we may colour it with one additional colour. This gives a colouring of $\overline{G}$ with $\chi(\overline{G}) = \omega(\overline{G})$.

All together we have shown that if G is perfect, P_G is two-level, and if P_G is two-level, then $P_{\overline{G}}$ is perfect. Here we see exactly the weak perfect graph theorem. Applying the result twice shows that G is perfect if and only if P_G is two-level.

4.4 Twisted Prisms of Independent Set Polytopes

Independent set polytopes of perfect graphs provide a situation where the twisted prisms of two-level polytopes are two-level. This fact was proven by Hansen [6].

The Hanson polytope is formed as the twisted prism of the independent set polytope. Throughout this section we will think of the independent set polytope of $G = ([n], E)$ as being embedded as $0 \times P_G$ in an $n+1$ dimensional vector space by adding an additional basis vector e_0. This convenient embedding allows us to describe the twisted prism construction. Indeed, the Hanson polytope is defined using a very specific embedding in $\mathbb{R}^{n+1}$.

Consider the linear transformation $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by

$$\varphi(x) = 2x - \mathbf{1} + 2e_0$$

This transformation scales, centers and translates the n-dimensional $0/1$ cube so that it is a ± 1 cube located at height 1. Taking the twisted prism of this transformed cube results in the full-dimensional ± 1 cube. Since independent set polytopes are $0/1$ polytopes the effect upon them is analogous. The Hanson polytope is the twisted prism of the independent set polytope after this transformation has been applied.

Definition 4.11 (Hanson Polytope)**.**
The Hanson polytope for a graph G is given by

$$H_G = \operatorname{conv}(\pm\varphi(P_G))$$

where P_G is the independent set polytope and φ is the transformation described above.

A more explicit description of the Hanson polytope is given by

$$H_G = \operatorname{conv}\left\{ \pm\left(-f_0 + 2\sum_{v_i \in S} e_i \right) : S \subseteq G \text{ independent} \right\}.$$

Here $f_0 = (-1, 1, \dots, 1)$. This form is useful for computations. The vertices in this description are points $\pm y_S$, labelled by independent subsets of the graph, where $y_S = -f_0 + 2\sum_{v_i \in S} e_i$. Note that the vertices of the graph are labelled by $v_1, \dots, v_n$ while the vector space is spanned by $e_0, e_1, \dots, e_n$.

The $n+1$ sets $\phi, \{v_1\}, \dots, \{v_n\}$ are independent sets in every graph G on n vertices. That is, there will be vertices corresponding to these sets in every Hanson polytope H_G. As a useful normalisation, we can consider the transformation $\theta : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which sends these $n+1$ vertices to the basis elements $e_0, \dots, e_n$. The image of a Hanson polytope under this transformation θH_G has some nice properties.

In particular because of the known vertices just mentioned, this normalised form guarantees that the dual, $(\theta H_G)^*$, is contained in the unit cube.

What do the vertices of the Hanson polytope look like under the transformation θ? For any $S \subseteq G$ we have a corresponding vertex $y_S = -f_0 + 2\sum_{v_i \in S} e_i \in H_G$. Under θ this vertex is

$$\theta(y_S) = \sum_{v_i \in S} e_i + (|S| - 1)e_0.$$

This explicit formulation will be vital in later calculations.

We are interested in the two-level property. In his paper Hansen [6] called his polytopes weak Hanner polytopes. For our purposes, following the discussion in Section 3.2 this is equivalent to two-level and centrally symmetric. Note that the weak Hanner polytope property is preserved by linear transformations and duality. Importantly, H_G is a weak Hanner polytope if and only if $(\theta H_G)^*$ is.

The main result of this section is that a Hanson polytope is two-level if and only if the graph G is perfect. Said result is similar to something we have seen before in theorem 4.2, namely that same conclusion holds for independent set polytopes. Indeed the previous result will be the basis of the proof here.

We will again use an integrality property, this time for the normalised Hanson polytope. Since the dual of the transformed Hanson polytope is contained within a cube, saying that it is integral is identical to saying that it is a 0/1 polytope. Here and later we will use the cube C^{n+1} in $\mathbb{R}^{n+1}$ with vertex coordinates of ± 1.

Lemma 4.3.
The Hanson polytope H_G is two-level if and only if $V((\theta H_G)^*) \subseteq V(C^{n+1})$.

Proof 4.3.
($\Rightarrow$) First assume that H_G is two-level, or equivalently that is is a WHP.

Since the empty set and singletons are all independent, $(\theta H_G)^* \subseteq C^{n+1}$. For each i, e_i is a vertex of θH_G. Thus

$$F_i = \{x \in (\theta H_G)^* : x \cdot e_i = 1\}$$

is a facet. Since H_G is a WHP, we know that $(\theta H_G)^* = \mathrm{conv}(F_i \cup -F_i)$. So for any vertex v of $(\theta H_G)^*$ and for all $i = 1, \ldots, n$, we have $v \cdot e_i) = \pm 1$. So $v \in V(C^{n+1})$.

($\Leftarrow$) Now assume that the vertices of H_G are contained in the cube.

Consider any facet F of $(\theta H_G)^*$ and the corresponding vertex $\theta(v)$ of θH_G. That is, $F = \{x \in (\theta H_G)^* : \theta(v) \cdot x = 1\}$. Suppose w is a vertex of $(\theta G_H)^*$. Then from the definition of F and the central symmetry we have $|\theta v \cdot w| \leq 1$. Since w is by assumption a vertex of a cube, and we know that v is labelled by some independent $S \subseteq G$ we have

$$(\theta v, w) = (|S|-1)w_0 + \sum_{v_i \in S} w_i.$$

This is an odd integer as $w_0, w_i = \pm 1$. So $(\theta(v), w) = \pm 1$ for all vertices of w. Thus $(\theta H_G)^*$ and hence H_G is a WHP.

Why are we so interested in this polytope $(\theta H_G)^*$? The next lemma demonstrates that it connects closely to the Hanson polytope of the complement graph.

Proposition 4.3.
H_G is a WHP if and only if $H_{\overline{G}} = (\theta H_G)^*$.

Proof 4.4.
($\Rightarrow$) First assume that H_G is a WHP.

Suppose S is an independent set in $\overline{G}$ and T is an independent set in G. Then x_S is a vertex of $H_{\overline{G}}$ and $\theta(x_T)$ is a vertex of θH_G. Now

$$\begin{aligned}(x_S, \theta x_T) &= \left(-f_0 + 2\sum_{v_i \in S} e_i, (|T|-1)e_0 + \sum_{v_r \in T} e_i\right) \\ &= -((|T|-1)(f_0, e_0) - \sum_{v_i \in T}(f_0, e_i) + 2\sum_{v_i \in S}\sum_{v_j \in T}(e_i, e_j) \\ &= |T| - 1 - |T| + 2\sum_{v_i \in S \cap T} 1 \\ &= 2|S \cap T| - 1\end{aligned}$$

Since an independent set and a clique can intersect at most in a single vertex we have $|(x_S, \theta x_T)| \leq 1$. This implies that x_S is contained in $(\theta H_G)^*$. That is, $H_{\overline{G}} \subseteq (\theta H_G)^*$.

Now since by assumption H_G is a WHP, by the previous lemma we know that the vertices of $(\theta H_G)^*$ are contained in the cube. That is, any vertex of $(\theta H_G)^*$ can be written as x_U for some vertex subset U of G. Is this set U independent in $\overline{G}$? If v_i, v_j are any two vertices in U, then

$$\begin{aligned}(\theta x_{\{v_i, v_j\}}, x_U) &= (e_0 + e_1 + e_j, x_U) \\ &= 3 \\ &\nleq 1\end{aligned}$$

So $x_{\{v_i,v_j\}}$ is not a vertex of H_G, so there is not an edge connecting them. In particular this means that the set U is independent in $H_{\overline{G}}$ and thus $(\theta H_G)^* \subseteq H_{\overline{G}}$.

Together these imply that $H_{\overline{G}} = (\theta H_G)^*$.

($\Leftarrow$) Now assume that $H_{\overline{G}} = (\theta H_G)^*$.

Since $(\theta H_G)^* = H_{\overline{G}}$ the vertices of $(\theta H_G)^*$ are contained in the cube. By the previous lemma this implies that H_G is a WHP.

By construction the top facet of the Hanson polytope corresponds to the independent set polytope. That is $\varphi P_G = \{x \in H_G : x_0 = 1\}$. The next lemma shows that a similar connection is true for the antiblocking polytope.

Lemma 4.4.
$\varphi \overline{P_G} = \{x \in (\theta H_G)^* : x_0 = 1\}$

Proof 4.5.
We consider P_G as being embedded in $\mathbb{R}^{n+1}$ so any point in P_G can be represented as $z = (0, z_1, \ldots, z_n)$. For such a z we have two conditions for z to be in $\overline{P_G}$. First we must have for all independent S

$$\left(z, \sum_{v_i \in S} e_i \leq 1 \right)$$

and second we must have

$$z_i \geq 0 \text{ for all } i.$$

If S is independent then,

$$\begin{aligned} \left(z, \sum_{v_i \in S} e_i \right) \leq 1 &\Leftrightarrow \sum_{v_i \in S} z_i \leq 1 \\ &\Leftrightarrow \sum_{v_i \in S} 2z_i \leq 2 \\ &\Leftrightarrow |S| - 1 + \sum_{v_i \in S} (2z_i - 1) \leq 1 \end{aligned}$$

Here we have $(\varphi z, \theta \alpha_S) = |S| - 1 + \sum_{v_i \in S} (2z_i - 1)$.

Furthermore

$$\begin{aligned} z_i \geq 0 &\Leftrightarrow 2z_i - 1 \geq -1 \\ &\Leftrightarrow (\varphi z, \theta \alpha_{\{v_i\}}) \geq -1 \end{aligned}$$

Since $\{v_i\}$ are all independent, we have $x \in \varphi \overline{P_G} \Leftrightarrow x \in (\theta H_G)^*$ and $x_0 = 1$.

With these three lemmas we can prove the main theorem of this section.

Theorem 4.3.
The Hanson polytope H_G is two-level if and only if G is perfect.

Proof 4.6.
In the previous section we saw that a graph G is perfect if and only if the independent set polytope is. Along the way this involved showing that G is perfect if and only if the antiblocking polytope $\overline{P_G}$ has integral vertices. The second lemma showed that $\varphi(\overline{P_G})$ is the top facet of $(\theta H_G)^*$. Since the bottom facet is just the reverse of the top, if $\overline{P_G}$ has integral vertices so too does $(\theta H_G)^*$. Together with the first lemma this demonstrated that H_G is a WHP if and only if G is perfect. We saw that for Hanson polytope the two-level and WHP properties are equivalent.

Both theorems in this chapter have a similar structure, namely they each show that a polytope is two-level if the graph the come from is perfect. Focusing on just the independent set polytopes of perfect graphs we can give an alternative interpretation. These polytopes provide a situation where the twisted prisms of two-level polytopes are two-level. In the next chapter we will explore another situation where the same phenomenon occurs.

Order Polytopes 5

Order polytopes are polytopes associated to posets. The structure of the order polytope, and indeed later on its twisted prism, is related to various combinatorial objects in the poset. As such, let's briefly review these definitions here.

Definition 5.1 (Poset)**.**
A poset $(P, \preceq)$ is a set of elements P together with a partial order $\preceq$.

Minimal (resp. maximal) elements in the poset are elements $x \in P$ such that there doesn't exists a $y \in P$, $y \neq x$, with $y \preceq x$ (resp. $x \preceq y$). Cover relations are the minimal relations required to define a poset. Two elements $x, y \in P$ form a cover relation if $x \preceq y$ and there is no $z \in P$, $z \neq x \neq y$, with $x \preceq z \preceq y$. In this section we will typically label the elements of the poset with the numbers $1, \ldots, n$. The five key poset structures we will use are chains, antichains, intervals, filters and linear extensions.

Definition 5.2 (Chain)**.**
A chain in a poset P is a collection of elements $C \subseteq P$ such that $x \preceq y$ or $y \preceq x$ for all $x, y \in C$.

Definition 5.3 (Antichain)**.**
An antichain is a poset is a set S of pairwise incomparable elements in a poset. That is, for all $x, y \in S$ we have

$$x \not\preceq y \text{ nor } y \not\preceq x.$$

A different way of describing a chain is a set of elements which are all comparable. In this way we see the connection between chains and antichains. Indeed, the definition here seems somewhat similar to the notion of cliques and independent sets. We will later exploit this resemblance in Section 6.2.

Definition 5.4 (Interval)**.**
An interval in a poset is comprised of all elements contained between two elements.

$$[a, b] := \{x \in P : a \preceq x \preceq b\}.$$

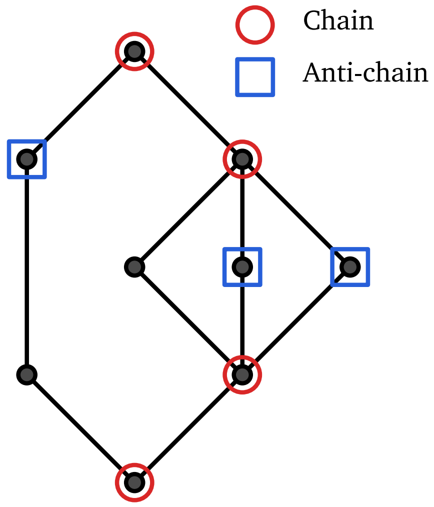

Figure 5.1: A chain and an antichain in an example poset.

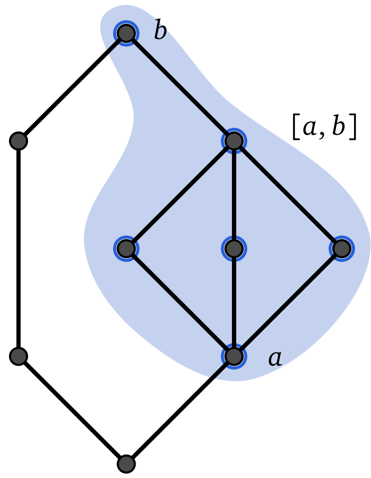

Figure 5.2: An interval in an example poset.

One way to think of an interval is as the union of all chains connecting the two elements.

A filter is a subset of the elements of a poset which is closed with respects to the partial order.

Definition 5.5 (Filter).
A subset $F \subseteq P$ is a filter if for all $a \in F$ and $b \in P$, if $a \preceq b$ then $b \in F$.

One particularly important class of filters are the principle filters. These are filters which are generated by a single element. They are best thought of as the upset of a single element. We write

$$P_{\geq x} = \{y \in P : x \preceq y\}$$

for the upset of x, with analogous definitions for $P_{>x}$, $P_{\leq x}$ and $P_{<x}$. Any filter can be expressed as the union of the upsets of all the minimal elements in the filter.

Akin to filters are the ideals of a poset. These are sets closed 'downwards' are always the complement of a filter. It is often convenient to forget the distinction between ideals and filters and to deal with the dividing line between filter and ideal, rather than the two sets themselves.

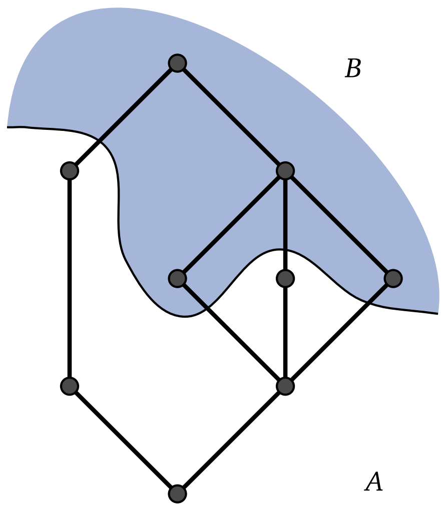

Figure 5.3: A filter in an example poset.

Definition 5.6 (Linear Extension)**.**
A linear extension of a poset $(P, \preceq)$ is a poset $(P, \leq)$ such that for all $x, y \in P$ we have

$$x \preceq y \Rightarrow x \leq y$$

and

$$x \leq y \text{ or } y \leq x.$$

Any relation satisfying the second property is called a total order on P.

When the elements of a poset are identified with dimensions in a linear space, the relations in the poset give rise to linear conditions. These restrictions form a polytope.

Definition 5.7 (Order Polytope)**.**
Given a poset $(P, \preceq)$ with n elements, $P = \{1, \ldots, n\}$, the order polytope $O(P)$

associated to the poset P is,

$$O(P) := \left\{ x \in \mathbb{R}^n : \begin{array}{ll} 0 \le x_i \le 1 & \forall\, 1 \le i \le n \\ x_i \le x_j & \forall\, i \preceq j \end{array} \right\}$$

The order polytope is an $\mathscr{H}$-polytope contained within the unit cube. There are very few low-dimensional order polytopes. In two-dimensions these are the triangle and the square and in three dimensions these are the tetrahedron, square based pyramid, triangular prism and cube.

Order polytopes are full dimensional. To see this, consider any extension of the poset to a total ordering. From this we obtain a full-dimensional simplex defined by $0 \le x_{i_1} \le \cdots \le x_{i_n} \le 1$ which is contained within the order polytope. In fact, this same technique allows us to triangulate the order polytope by considering all linear extensions of the poset. The details of this triangulation are given and built upon in the next chapter.

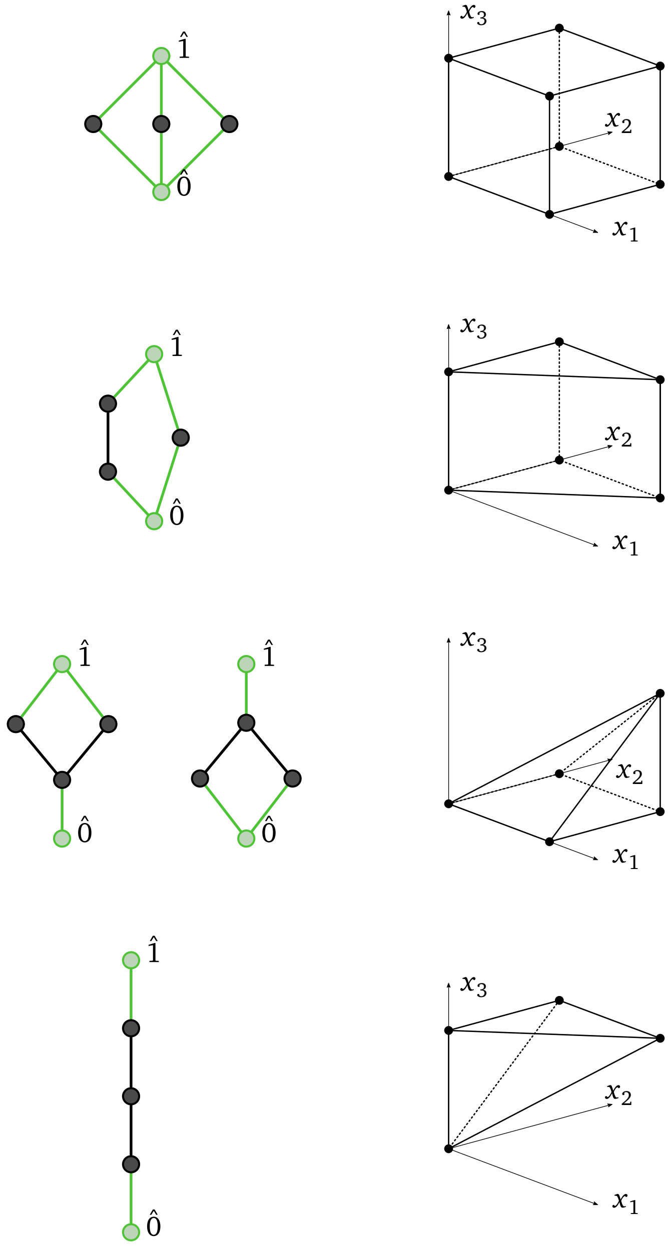

Figure 5.4: The three-dimensional order polytopes.

5.1 Structure of the Order Polytope

Of the inequalities given in the definition of the order polytope, not all are essential. That is, not all are facet defining inequalities. The necessary inequalities in the polytope are exactly those required to define the poset. These come from the minimal elements, maximal elements and cover relations.

Proposition 5.1 (Facets of the Order Polytope)**.**
The facets of the order polytope are given by,

$$
\begin{aligned}
0 \leq x_i &: x_i \text{ is minimal in } P \\
x_i \leq 1 &: x_i \text{ is maximal in } P \\
x_i \leq x_j &: i \prec j \text{ is a cover relation in } P.
\end{aligned}
$$

To simplify the description of the facets of the order polytope we can introduce an extended poset $\hat{P}$ by adding global minimum and maximum elements to P. That is, elements $\hat{0}$ and $\hat{1}$ with $\hat{0} \preceq p \preceq \hat{1}$ for all $p \in P$. The facets of the order polytope $O(P)$ then correspond exactly to the cover relations in this extended poset $\hat{P}$. Throughout the rest of this section, it will be assumed that we refer to the extended poset.

Faces of the Order Polytope

The combinatorial description of the other faces of the order polytope builds upon the identification of facets with cover relations. One way to view the correspondence between facets and cover relations is as collapsing the relation. The cover relation $i \preceq j$ tells us that the polytope only contains points x for which $x_i \leq x_j$. The corresponding facet is obtained by setting $x_i = x_j$.

Since any face can be written as an intersection of facets, to determine the faces of the order polytope we must simply select which cover relations (of the extended poset) to collapse. There is one type of nonsensical selection we have to avoid. Namely, when two elements of the poset are identified by collapsing cover relations, all cover relations between them must be collapsed. Examples of what can go wrong here can be found in Figure 5.5.

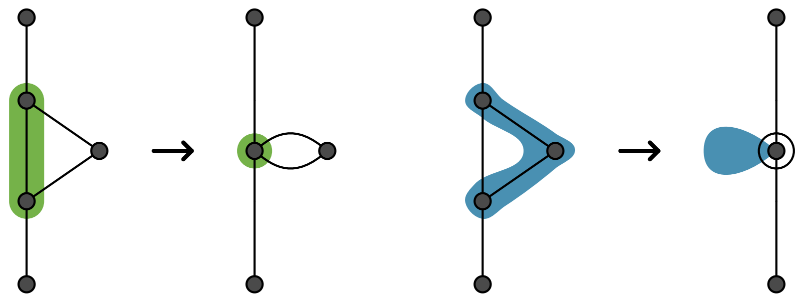

Figure 5.5: Examples of selections of cover relations to collapse which do not result in a valid face of the order polytope.

In more standard language we don't choose cover relations to collapse, but rather subdivide the elements and create a new poset with an induced ordering. Here our sensible selections of cover relations corresponds to a subdivision of the extended poset that is connected, closed, compatible and convex. Such a partition of the elements of a poset is called a face partition [12].

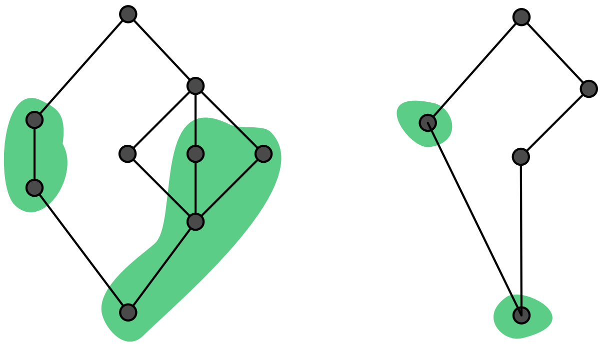

Figure 5.6: A partition of the elements in a poset and the accompanying collapsed poset.

The previous combinatorial description of the faces of the order polytope reduces to a very simple way of describing the vertices. In a vertex the coordinates must be either zero or one. This corresponds to every element of the poset being identified with either the global minimum or maximum. Such a selection of cover relations is precisely a filter in the poset. Intuitively, such a filter can be thought of as a horizontal line through the Hasse diagram of the poset, dividing it into a top and bottom section. Then the vertex corresponding to a filter is simply the characteristic function of the filter.

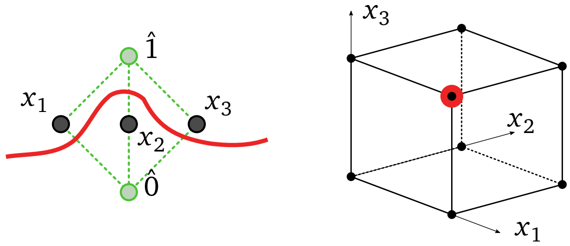

Figure 5.7: An example of a filter and the corresponding vertex.

Since the combinatorial description of the faces of the order polytope is so crucial to the proof that order polytopes are two-level, it is worth developing some intuition for the structures by studying the triangular prism as a detailed example in Figure 5.8.

Optimising over an Order Polytope

Identifying vertices of the order polytope with filters gives us a convenient way to think about optimising over the order polytope. Suppose we wish to optimise over an order polytope in the direction $u = (u_1, \ldots, u_n)$. To do this, we can label

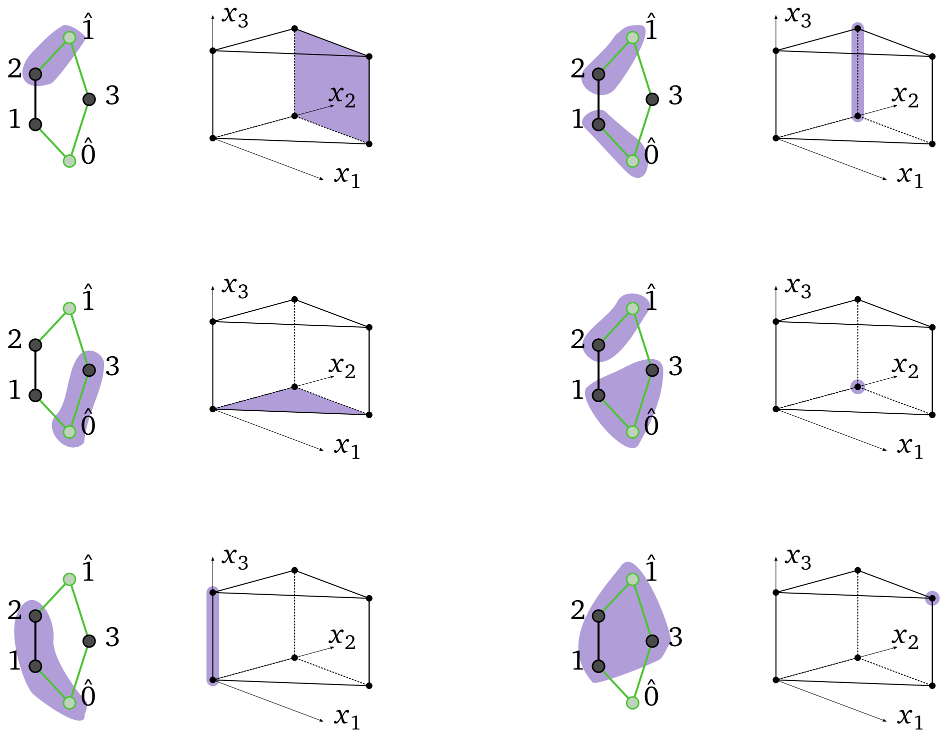

Figure 5.8: Poset partition in the triangular prism.

each element i of the poset by the corresponding u_i. Then the optimisation is equivalent to finding the filter which maximises the sum of the u_i corresponding to those elements identified with one. That is,

$$\max_{F \subseteq P} \sum_{i \in F} u_i$$

where the maximum is taken over all filters F of the poset. An example of this sort of optimisation is shown in Figure 5.9.

In general this optimisation is a difficult calculation, and indeed the technique proposed here would be much less efficient that the standard algorithms. However, if the direction u is special we can reason through the answer directly. For example, if u is the normal vector of the facet corresponding to a cover relation $i \preceq j$, then optimisation in the direction u will occur for exactly those filters which don't separate the two endpoints of the cover relation. This what we would expect given the description of the faces of the order polytope as selections of cover relations to collapse. On the other hand, optimising in the direction $-u$ gives all the filters that do pass through the cover relation $i \preceq j$. Together this describes all filters and hence all vertices, showing that order polytopes are two-level. The same technique will be used later to show that the twisted prisms of order polytopes are two-level.

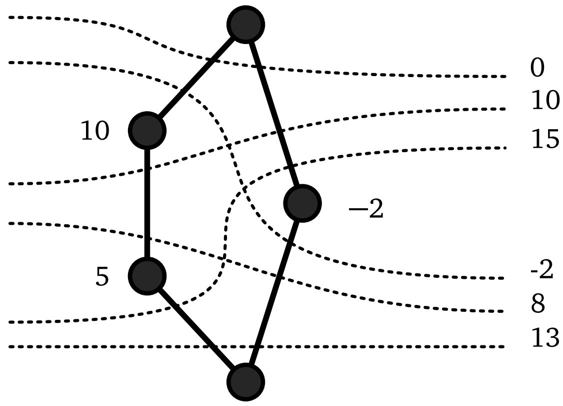

Figure 5.9: Optimisation over the order polytope.

We just saw with a convoluted technique that order polytopes are two-level. However, there is a much simpler and more direct proof. Consider the values each of the inequalities in the definition can take at integer points. The first class of inequalities, $0 \leq x_i \leq 1$, can only take values 0 or 1 on the vertices of the cube. For the second class of inequalities, $x_i - x_j \leq 0$, the two levels are given by $x_i - x_j = -1$ and $x_i - x_j = 0$.

Observe that all two-level polytopes are not necessarily order polytopes. One obvious counterexample is the octahedron. This three-dimensional polytope is two-level but is not an order polytope (see Figure 5.4).

5.2 Twisted Prisms of Order Polytopes are Two-Level

Recall that Hansen's proof in Chapter 4 demonstrated that the twisted prisms of independent set polytopes of perfect graphs are two-level. One might hope that a similar result is true for the twisted prisms of order polytopes.

The structure of the twisted prism of order polytopes is not trivial. For example, the twisted prisms of order polytopes are not always order polytopes. A counterexample here is provided by the triangle. This is an order polytope but its twisted prism, the octahedron, is not. A slightly more general question we considered is whether the twisted prisms of order polytopes are alcoved polytopes. These are polytopes which have facet normal vectors corresponding to roots of an irreducible root system. This is again not true. A proof of this is given later for the chain polytope and applies equally well to the twisted prism of the order polytope.

The twisted prisms of order polytopes are indeed two-level. To prove this fact, we will first need to explore a general tool to describe the facets of general twisted prisms.

Theorem 5.1 (Twisted Prisms of Order Polytopes)**.**
The twisted prisms of order polytopes are two-level.

Facets of Twisted Prisms

To prove that the twisted prisms of order polytopes are two-level we must first examine a little the structure of twisted prisms in general. For a polytope to be two-level, we must have two levels in the direction of every facet. What are the facets of a twisted prism? We know of two obvious facets at the top and bottom which have the form P and $-P$. The other facets all occur stretched between the top and bottom, by which we mean there are vertices in both. Clearly the top and bottom facets satisfy the two-level property as the construction of the twisted prisms demonstrates that they are each the other's second level. To test whether a twisted prism is two-level, we must then focus on the facets stretching from top to bottom. Here we refer to such facets (and similarly faces) as non-trivial facets.

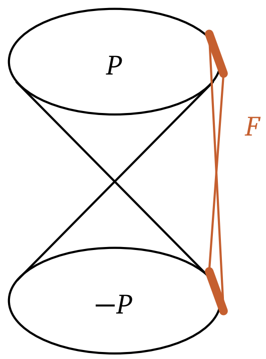

Figure 5.10: Facets of a twisted prism.

How might we describe these non-trivial facets? Faces of a polytope are defined by supporting hyperplanes. Such supporting hyperplanes for a non-trivial face will intersect the top and bottom levels of the twisted prism in hyperplanes of one dimension smaller. We can identify the top and bottom levels of the twisted prism with $\mathcal{O}(P)$ and $-\mathcal{O}(P)$. This results in a pair of distinct, parallel, supporting hyperplanes for P. Such a pair is called a sandwiching pair of hyperplanes, for obvious reasons.

The next question is which of these sandwiching pairs of hyperplanes correspond to facets and not just lower dimensional faces. One defining characteristic of the supporting hyperplanes of facets is that they are defined rigidly. That is to say, the cone of normal vectors for supporting hyperplanes of a facet is one-dimensional. This rigidity is preserved when we project from supporting hyperplanes of non-trivial facets in the twisted prism to sandwiching pairs in the original polytope. In the projection rigidity means that given the specific vertex incidences of the two sandwiching hyperplanes and the requirement that they are parallel, there

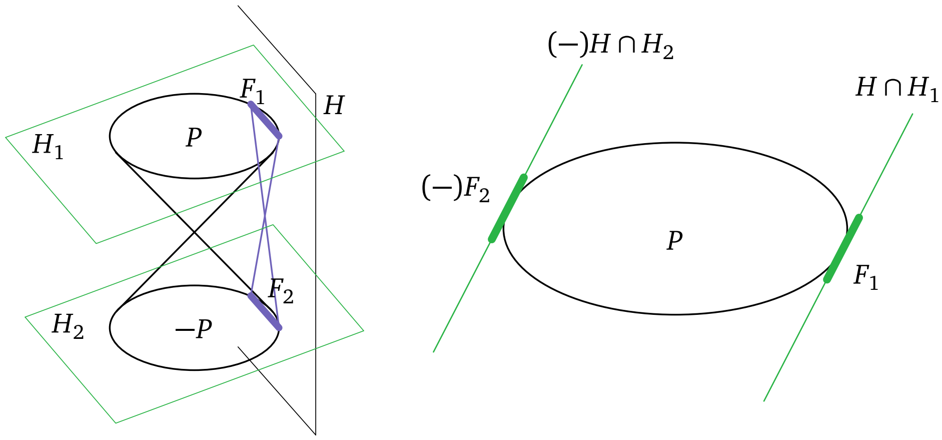

Figure 5.11: Projection to sandwiching hyperplanes.

is a unique (up to scaling) normal vector for the two parallel hyperplanes. The two-level property of a twisted prism is then equivalent to saying that every rigidly defined pair of sandwiching hyperplanes contain all vertices.

This may seem like a tortured translation of the problem, but it allows us to explore the facets of the twisted prism by directly studying the base polytope.

> **Proposition 5.2** (Two-Level Twisted Prisms)**.**
> The twisted prism $\mathscr{T}(P)$ of a polytope P is two-level if and only for every pair H_1 and H_2 of sandwiching hyperplanes of P which are rigidly defined, every vertex of P is contained in either H_1 or H_2.

If P is two-level, then we already know something about some of the non-trivial facets of the twisted prism. Take any facet F of P. Then there is a second level F' in P containing all other vertices. In the twisted prism there will be two facets associated to F, $\pm\mathrm{conv}\big(F \times 1, -F' \times -1\big)$. Between them the two facets contain all the vertices of the twisted prism. That is, for these facets the two-level property holds. Unfortunately these are not the only type of non-trivial facets which can occur in a twisted prism. Two lower-dimensional faces in the base polytope can combine to form a facet of the twisted prism. The simplest example of this phenomenon occurs with the tetrahedron. Here there are facets in the twisted prism corresponding to the tetrahedron itself (2), the facets of the tetrahedron (8) and opposite pairs of edges (6). In a low-dimensional example like the tetrahedron we can directly observe that the face pairs result in a rigidly defined sandwiching pair of hyperplanes.

Sandwiching Hyperplanes and Optimisation

One way to consider a supporting hyperplane of a polytope is as a result of an optimisation process in a given direction. For any vector u we can define a pair of

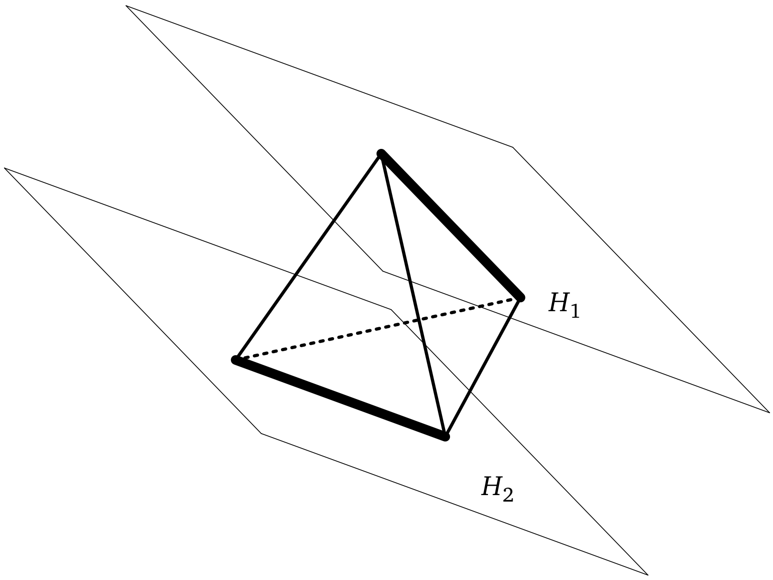

Figure 5.12: Sandwiching hyperplanes in the tetrahedron.

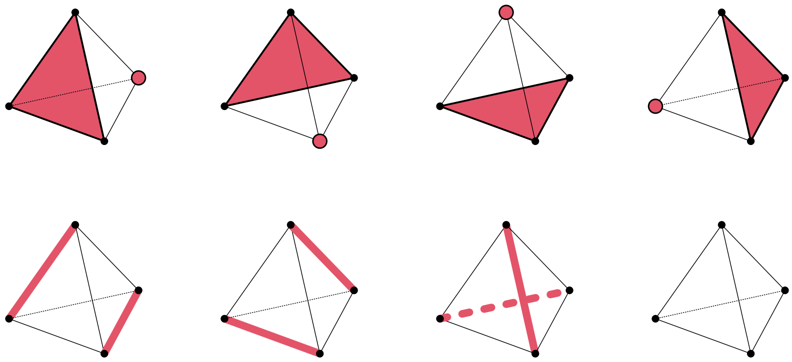
Figure 5.13: Facet defining pairs in the tetrahedron.

sandwiching hyperplanes by optimising in the directions u and $-u$. Reversing this process, we can take two faces F_1 and F_2 of P and ask what are the vectors u for which optimising in the directions u and $-u$ gives the faces F_1 and F_2? Rigidity for sandwiching hyperplanes is then the requirement that, for a specific pair of faces, the directions u (and $-u$) which optimise on these faces form a one-dimensional space.

Definition 5.8 (Rigidity).
A pair of faces F_1 and F_2 in an order polytope $O(P)$ form a rigid pair if the space of vectors u with $O(P)^u = F_1$ and $O(P)^{-u} = F_2$ is one-dimensional.

Thinking now about the order polytope, we can determine which pairs of faces give rise to rigid sandwiching hyperplanes.

Optimising in Facet Directions

We know that order polytopes are two-level, so any facet of the twisted prism which arises from connecting a facet in the order polytope with its second level satisfies the two-level property. This fact is easy to see in the language of sandwiching hyperplanes and optimisation. Suppose F is a facet of the order polytope $O(P)$. Then F corresponds to a cover relation in the extended poset $\hat{P}$ and has normal vector $u = e_i - e_j$ for some $i \prec j$. Since this normal vector has so few non-zero coordinates, it is easy to directly observe in the poset how to optimise in the directions of u and $-u$, for example in Figure 5.14.

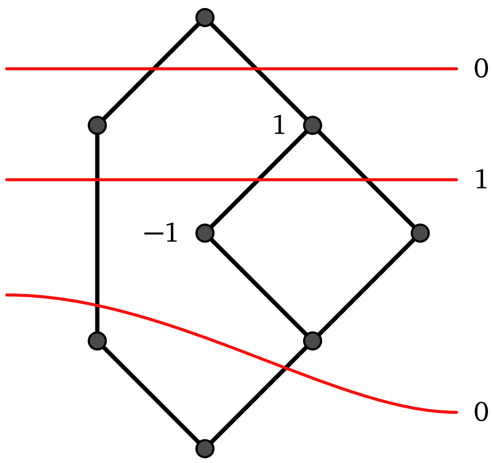

Figure 5.14: Optimising in the direction of a facet.

The two directions each optimise on a set of vertices, corresponding to the filters which either separate i and j or don't. Going the other way, we can see that we will only obtain these two sets of vertices when the direction of optimisation is some scaling of the normal vector u.

Optimising for Intervals

There is a second situation which mimics optimising over a facet. This occurs when one face in the rigid pair arises by collapsing an interval in the poset. One half of the rigid pair will be given by all filters which contain none of the interval or all of it and the other by all filters which separate the endpoints.

Consider an interval $[i, j]$. If u optimises on all filters which do not contain j and all filters which do contain i then we must have $u_j < 0$, $\sum_{k \in [i,j]} u_k = 0$ and $u_k = 0$ for all $k \notin [i, j]$. If $-u$ optimises on all filters which contain j but not i we must have $u_k = 0$ for all $i \prec k \prec j$. This then give $u_i = -u_j$.

This is a one-dimensional space of normal vectors and hence the pair of faces is rigid.

Connected Components

We saw that faces arising from collapsing cover relations in an interval in $\hat{P}$ form rigid pairs. Indeed, all connected components in the face partition of half a rigid pair must be intervals.

> **Proposition 5.3.**
> If F is half of a rigid pair of faces in $\mathscr{O}(P)$ and the face partition of $\hat{P}$ defining F has a single non-trivial component, then that component in an interval.

Consider a face F of the order polytope arising from collapsed cover relations in $\hat{P}$ which form a single connected component. Equivalently, a face partition with only a single non-trivial connected component M. Consider any u which optimises on F. That is, u attains its maximum on all filters which either contain all of M or none of it. Then u must be negative for the maximal elements of M and positive for the minimal elements of M.

Now $-u$ will optimise on some subset of the filters which contain all the maximal elements of M and none of the minimal elements. The precise subset will depend on the values of u for the elements in M which are neither minimal nor maximal. Importantly, this subset does not depend on the exact values associated to the minimal and maximal elements. Thus if there is more than one minimal or maximal element in M, the dimension of the space of functionals u defining any sandwiching pair including F is greater than one.

Thus every non-trivial connected component of a face partition which describes one half of a rigid pair has a unique minimal and maximal element. That is, it is an interval.

We can extend this same idea to find the nature of all face partitions leading to rigid pairs.

Combining Connected Components

We saw that face partitions with a single component must be intervals. How then can these intervals combine to form more complicated face partitions? As an illuminating analogy, let us borrow from electronics. Two intervals can either be connected in series or parallel. It will turn out that all the intervals must connect in series, forming a chain of intervals.

> **Proposition 5.4.**
> If F is half of a rigid pair of faces in $\mathscr{O}(P)$ and the face partition of $\hat{P}$ defining

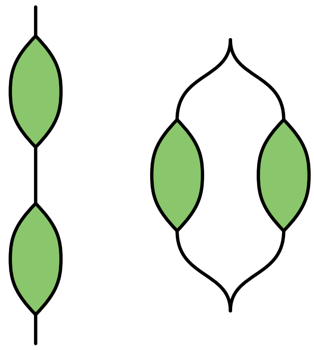

Figure 5.15: Connecting intervals in series or parallel.

> F is comprised of non-trivial connected components $M_1, \ldots, M_k$, then every M_k must be an interval and the endpoints of the M_i must form a chain in $\hat{P}$.

Suppose that a face F of the order polytope is one half of a rigid pair and is described by a face partition with non-trivial connected components $M_1, \ldots, M_k$. Any function U which optimises on F must have negative values on all maximal elements of the M_i and positive values on all minimal elements of the M_i. Let A be the set of globally minimal elements amongst the M_i and B the set of globally maximal elements amongst the M_i. Then any filter which optimises in the direction $-u$ must contain all the elements in B and none of the elements in A. Again, the precise set of filters upon which $-u$ optimises will not depend on the values that u takes on the elements in A and B. Thus, if there is more than one global minimal or maximal element amongst the face partition the face F cannot belong to a rigid pair.

There is then some component M_i with $m_i \in M_i$ maximal amongst the face partition. Fixing the values of u on M_i We can then apply the same argument to the remaining components to again get a single maximum. Therefore, all the components are intervals and all of their endpoints are comparable.

In our analogy, this means that all the intervals occur in series.

Rigid Sandwiching Pairs

Conditions for a pair of posets to be rigid is then that each face is described by a well ordered sequence of intervals in the poset.

That is, each face in a rigid pair arises by collapsing the cover relations in a sequence of intervals

$$[i_0, i_1], [i_2, i_3], \ldots, [i_{m-1}, i_m]$$

where $i_0 \prec i_1 \prec \cdots \prec i_m$.

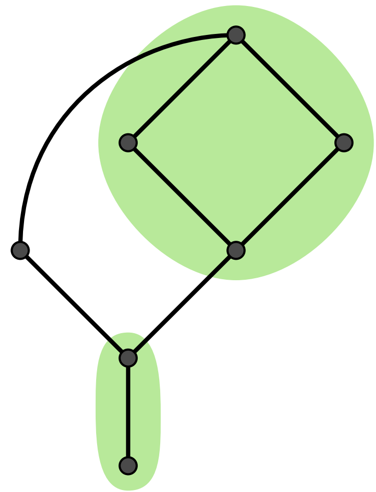

Figure 5.16: Rigid pairs in a poset.

The Two-Level Property

Having described all the facets of the twisted prism of an order polytope, we can easily check that the twisted prism of an order polytope is two-level. Every facet, with the exception of the top and bottom copies of $\mathscr{O}(P)$, connects two faces of the order polytope. These two faces are described by an alternating sequence of intervals in the poset, extending between the artificial minimum and maximum elements. Since every filter must pass through this chain at some point, every vertex of the order polytope is contained within one of the two faces in the rigid pair. Projecting back to the twisted prism then demonstrates that the corresponding mixed facet satisfies the two-level property.

5.3 Full Facet Structure of the Twisted Prisms of Order Polytopes

Following from the previous proof we have know all the facets of the twisted prism of an order polytope. These facets are either the top or bottom copies of $O(P)$ or arise from a rigid pair of faces in $O(P)$. The rigid pairs correspond to an alternating sequence of intervals in the poset. Instead of describing this sequence by the intervals, we may instead look at the transition points between the intervals. These transition points must form a chain in the poset.

This gives rise to a very elegant description of the facets of the twisted prism of an order polytope. Every chain in the poset P corresponds to two facets in $\mathscr{T}(O(P))$. These facets are the negatives of each other and come from connecting a face in $O(P)$ and a face in $-O(P)$. Given a chain $\{i_0, i_1, \ldots, i_m\}$ in the poset the

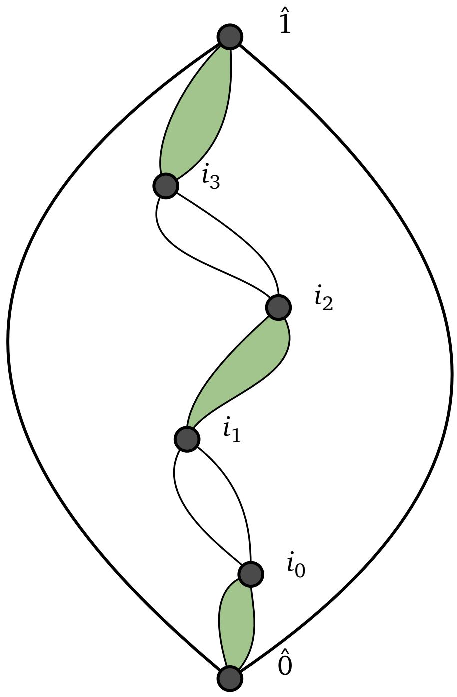

Figure 5.17: Intersection points of a rigid face pair.

two connected faces come from collapsing the cover relations in

$$[\hat{0}, i_0], [i_1, i_2], \ldots, [i_{m-1}, i_m]$$

and

$$[i_0, i_1], \ldots, [i_m, \hat{1}]$$

with an obvious variation depending on whether m is even or odd (above m is odd). Note that if we are happy to treat the empty set as a chain then this corresponds to the copies of $O(P)$ and $-O(P)$.

5.4 Facet Count of Twisted Prisms of Order Polytopes

Previously we obtained a description of the facets of the twisted prisms of order polytopes. Using this result we can give an explicit formula for the number of these facets. From the description, we know that the number of facets in the twisted prism is twice the number of chains in the poset. Using an incidence matrix for the poset we can calculate the number of chains.

Definition 5.9 ((Poset) Incidence Matrix)**.**

For any poset P the incidence matrix I_P is defined by

$$(I_P)_{i,j} = \begin{cases} 1 & i \prec j \\ 0 & \text{otherwise.} \end{cases}$$

Note that the inequality is strict, although we could easily rewrite the formula to use a non-strict incidence matrix instead. If we chose to order the elements of the poset with a linear extension, the incidence matrix will be upper triangular and hence we see that is is nilpotent.

How does this incidence matrix relate to chains? Suppose we want to count the number of chains from a to b of length two, here written as $C^2_{a,b}$. This is exactly one if $a \prec b$ and zero otherwise, i.e. $(I_P)_{a,b}$.

$$C^2_{a,b} = (I_P)_{a,b}.$$

Similarly the number of length three chains, $C^3_{a,b}$, can be counted by looking at all the possible intermediate stops. That is,

$$C^3_{a,b} = \sum_{i=1}^{n} C^2_{a,i} \cdot C^2_{i,b} = \sum_{i=1}^{n} (I_P)_{a,i} \cdot (I_P)_{i,b} = (I_P^2)_{a,b}.$$

This same pattern continues giving a formula for the number of chains of length m between any two elements.

$$C^m_{a,b} = (I_P^{(m-1)})_{a,b}.$$

Given that we have ordered the elements of the extended poset $\hat{P}$ with a linear extension the incidence matrix is upper triangular. In particular we know that the matrix $1 - I_{\hat{P}}$ is invertible. Thus the total number of chains in P is given by,

$$\begin{aligned} C &= (I_{\hat{P}})_{\hat{0},\hat{1}} + (I^2_{\hat{P}})_{\hat{0},\hat{1}} + (I^3_{\hat{P}})_{\hat{0},\hat{1}} + \dots \\ &= (I_{\hat{P}} + I^2_{\hat{P}} + I^3_{\hat{P}} + \dots)_{\hat{0},\hat{1}} \\ &= \left(1 \cdot (1 - I)^{-1}\right)_{\hat{0},\hat{1}} \end{aligned}$$

This is a relatively simple formula that only requires one (sparse) matrix inverse to be computed.

5.5 Face Structure of the Twisted Order Polytope

Given that we know the complete set of vertices, facets and corresponding inclusion relations we should a priori know the complete face description of the twisted order polytope. Indeed for any one example, it is relatively simple to calculate the complete list of faces from the poset alone. That said, we have been unable to come up with an elegant combinatorial description of the faces. However these is a nice description for the edges.

Edges of the Twisted Order Polytope

We can combinatorially describe the edges of the twisted prism of an order polytope. The edges of the order polytope are formed by pairs of filters F, F' with $F \subseteq F'$ such that the difference $F' \setminus F$ is connected. These will also be edges of the twisted prism. The remaining edges connect one vertex in the copy of $\mathcal{O}(P)$ and one in the copy of $-\mathcal{O}(P)$. Here we can alternatively think of one vertex as corresponding to a filter and the other an ideal.

Proposition 5.5 (Vertical Edges of the Twisted Prism)**.**
Given a filter $F \subseteq P$ and ideal $I \subseteq P$, the corresponding vertices form an edges of $\mathcal{T}(\mathcal{O}(P))$ if and only if

$$\forall i \in I : P_{\geq i} \cap (P \setminus (I \Delta F)) \neq \emptyset$$

and

$$\forall f \in F : P_{\leq f} \cap (P \setminus (I \Delta F)) \neq \emptyset.$$

Proof 5.1.
The direct proof of this fact given here uses the same transformation as in the proof that the twisted prism is two-level. Vertical faces in the twisted prism correspond to the intersections of sandwiching pairs of hyperplanes in the order polytope. This will be an edge precisely when both hyperplanes intersect the order polytope in a single vertex. That is, two vertices v_1, v_2 in the order polytope will form an edge in the twisted prism if and only if there is a vector u such that

$$\mathcal{O}(P)^u = v_1 \text{ and } \mathcal{O}(P)^{-u} = v_2.$$

Observe that optimising $-u$ over filters is the same as optimising u over ideals.

($\Leftarrow$)

Suppose that the condition given above is true for some ideal I and filter F. Consider the partition of the poset into the sets $I \cap F$, $I^C \cap F^C$, $P_{>I\cap F}$, $P_{<I\cap F}$, $F \setminus I \setminus P_{>I\cap F}$ and $I \setminus F \setminus P_{<I\cap F}$. Take $\epsilon > 0$ small and define a functional u by

$$u(x) = \begin{cases} x \in I \cap F & 1 \\ x \in I^C \cap F^C & -1 \\ c \in P_{>I\cap F} & -\epsilon \\ c \in P_{<I\cap F} & -\epsilon \\ \text{otherwise} & \epsilon \end{cases}$$

Then this u optimises uniquely over filters on F and over ideals on I. To see this note that both F and I contain all the points which take value 1 and none of the points which take value -1. Thus any filter or ideal which was optimal would have to do this. Furthermore by the assumed condition, adding a single extra element with value ϵ to the filter (or ideal) would necessitate adding an element with value -1. Therefore the given filter and ideal span an edge in the twisted prism.

($\Rightarrow$)

Suppose the given condition doesn't hold for a filter F and ideal I. Then w.l.o.g. there exists some $i \in I \setminus F$ such that $P_{\geq i} \subseteq I \Delta F$. Write

$$I' = I \setminus P_{\geq i}$$

and

$$F' = F \cup P_{\geq i}.$$

These are also an ideal and filter respectively. Write X for the moved set. That is,

$$X = I \setminus I' = F' \setminus F.$$

Now, given any $u \in \mathbb{R}$, consider the value the functional u takes on X. If this is positive then u takes a larger value on F' than F. If the value is negative then u takes a larger value on I' than I. Finally if u is zero then it takes the same value of F and F', so cannot optimise uniquely on F. That is, there cannot exist a vector $u \in \mathbb{R}$ such that the linear functional optimises uniquely on F and I. So the filter and ideal do not span an edge.

5.6 Duality

In Hansen's proof that the twisted prism of an independent set polytope for a perfect graph was two-level the key insight was to observe that after a clever linear transformation, the dual of the twisted prism was the Hansen polytope of the complement graph. The proof that twisted prisms of order polytopes given here uses a different technique. However there is a similar notion at play. To see this we will have to introduce the valuation polytope.

In a poset the join of two elements, written $x \vee y$, is defined as the unique smallest element greater than both x and y. Similarly the meet $x \wedge y$ is the unique largest element smaller than both. A poset where these two objects always exist is called a lattice. One motivating example is the powerset ordered by inclusion where the meet and join are given by union and intersection respectively. The aptly named face lattice of a polytope is also a lattice in the sense described here.

In the context of poset polytopes, one important lattice is the lattice of ideals, $I(P)$. Here the order is inclusion and meet and join are also given by union and intersection. (Note that we could alternatively use the lattice of filters.)

A valuation $f : L \to \mathbb{R}$ on a lattice L is a function which satisfies

$$f(x \vee y) = f(x) + f(y) - f(x \wedge y).$$

As an analogy which motivates the definition, consider area of sets.

The valuation polytope is formed by all valuations which take values between zero and one.

Definition 5.10 (Valuation Polytope)**.**
The valuation polytope $V(P)$ is given by

$$V(P) = \{f \in \mathbb{R}^{V(P)} : 0 \leq f \leq 1, f \text{ is a valuation on } L(P)\}$$

Valuations on the ideal lattice correspond to functionals on the poset. For any $u \in \mathbb{R}^P$, write v_u for the valuation given by

$$v_u(I) = \sum_{i \in I} u(i).$$

Then the valuation polytope can be embedded in $\mathbb{R}^n$ as

$$V(P) = \{u \in \mathbb{R}^n : 0 \leq v_u \leq 1\}.$$

Note that while both the order and valuation polytopes arise from a poset there seems to be little to connect them. In [3] Dobbertin shows that the vertices of the valuation polytope are expressed by a familiar set of vectors.

Proposition 5.6 (Vertices of the Valuation Polytope)**.**
The vertices of the valuation polytope are integer valuations corresponding to functionals h_C where C is a chain in P, $C = \{c_0, c_1, \ldots, c_m\}$, and h_C is given by

$$h_C(x) = \begin{cases} 0 & \text{for } x \in P \setminus C \\ (-1)^i & \text{for } x = c_i. \end{cases}$$

The vectors h_C described above are exactly half of the facet normals of the twisted prism of the order polytope (without the height coordinate). Indeed, up to a linear transformation, the dual of the twisted prism of the order polytope is the twisted prism of the valuation polytope. The analogy with Hansen's proof should be clear.

Proposition 5.7 (Order and Valuation Polytopes)**.**
The twisted prisms of the order and valuation polytopes are, up to some convenient embedding, duals.

We can write the valuation polytope as the convex hull of its vertices,

$$V(P) = \operatorname{conv}\{h_C : C \subseteq P \text{ chain}\}.$$

A different, but related polytope can be obtained by slightly modifying the vertices. For any chain C, we use the length of the chain as a parity on the functional,

$$h'_C = (-1)^m\, h_C = \begin{cases} 0 & \text{for } x \in P \setminus C \\ (-1)^{(i+m)} & \text{for } x = c_i. \end{cases}$$

We can define the modified $V'(P)$ by

$$V'(P) = \operatorname{conv}\{h'_C : C \subseteq P \text{ chain}\}.$$

Why would we wish to change the vertices in this way? Consider the linear functional over the valuation polytope given by the vector of ones. This function takes value 0 on vertices corresponding to chains where m is odd and 1 when m is even. Considering this same vector over the twisted prism of the valuation polytope we see that it again takes two values on all the vertices. Furthermore this functional equally divides the vertices in two. As such, the twisted prism of the valuation polytope is affinely equivalent to the twisted prism of the modified polytope.

$$\mathcal{T}(V(P)) \simeq \mathcal{T}(V'(P)).$$

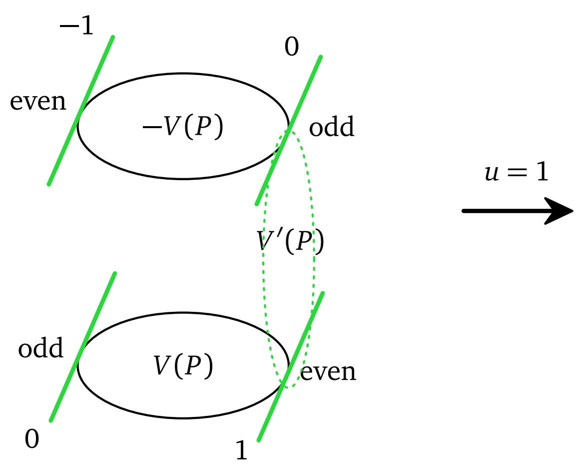

Figure 5.18: Two levels in the twisted valuation polytope.

We then consider a specific embedding of the twisted prism of the valuation polytope, described by giving an embedding of the modified polytope.

$$\mathcal{T}(V(P)) \simeq \operatorname{conv}\left(\pm\left(\tfrac{1}{2} \times V'(P)\right)\right)$$

Embedding the twisted prism of the order polytope as

$$\mathscr{T}(\mathscr{O}(P)) = \mathrm{conv}(\pm(1 \times 2\mathscr{O}(P)))$$

we can explicitly compute the facet normals. Note that this embedding of the twisted prism contains the origin. Facets in the twisted prism arise from the two levels in the order polytope obtained by optimising in the directions h_C. The two levels are either 0 and -1 if m is even or 1 and 0 if m is odd.

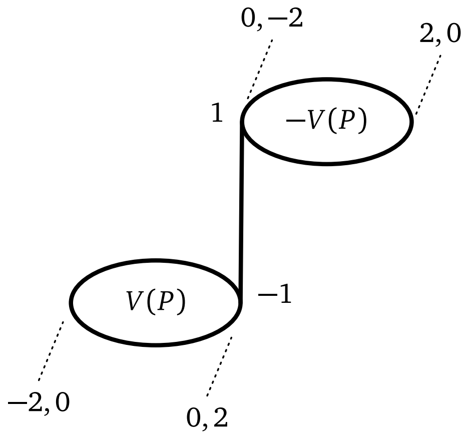

Figure 5.19: Normal vectors of the twisted order polytope.

Adjusting for the height coordinate then gives the following set of inequalities for the twisted order polytope,

$$\left\{x \in \mathbb{R}^{n+1} : \pm(-1^m, h_C) \cdot x \leq 1 : C \text{ chain in } P\right\}.$$

This set of normal vectors is equivalent to

$$\{\pm(1, (-1)^m h_C) : C \text{ chain in } P\}$$

which are the vertices of $\mathscr{T}(V'(P))$ from before.

Thus the dual of the twisted order polytope is isomorphic to the dual of the twisted valuation polytope.

Triangulations 6

We observed earlier in Chapter 3 a connection between two-level polytopes and triangulations. Namely, that two-level polytopes are also compressed. Recall that compressed means that every simplex in a pulling triangulation has the minimum simplex volume in the lattice which contains the polytope. This has some nice consequences. For example, since two-level polytopes are 0/1, the minimal volume is $\frac{1}{d!}$. In this way, counting the number of simplices in a pulling triangulation of a compressed polytope automatically gives us a measure of its volume.

There is a well known standard triangulation of the order polytope. In this section we extend this triangulation to a triangulation of the twisted prism and explore the consequences of a connection between the order polytope and the independent set polytope.

6.1 Standard Triangulation of the Order Polytope

The order polytope has a very simple standard triangulation arising from linear extensions of the poset. A total order of n elements, $i_1 \preceq i_2 \preceq \cdots \preceq i_n$, corresponds to an n-simplex in the unit cube. This simplex is expressed as

$$\{x \in \mathbb{R}^n : 0 \leq x_{i_1} \leq \cdots \leq x_{i_n} \leq 1\}.$$

The set of such simplices obtained by considering every total order of n elements is a triangulation of the unit cube in $\mathbb{R}^n$. Each simplex has volume $\frac{1}{n!}$ and there are $n!$ total simplices.

The triangulation of an order polytope then follows by selecting the simplices above which are contained within the order polytope. These will be exactly those simplices whose total orders are compatible with the poset, that is, linear extensions.

Proposition 6.1 (Triangulation of the Order Polytope).
Given a poset P, the order polytope $\mathcal{O}(P)$ is triangulated by the simplicies

$$0 \leq x_{i_1} \leq \cdots \leq x_{i_n} \leq 1$$

for all linear extensions $i_1 \preceq \cdots \preceq i_n$ of the poset P.

Since the linear extension are by definition compatible with the poset, it is clear that each simplex in the triangulation is contained in the order polytope. On the other hand, any point in the order polytope is contained in one of these simplices. To see this, consider any generic point in the order polytope. That is, a point where no two coordinates are the same. Then the coordinates of the point induce a total order on the n dimensions. This total order must be a linear extension since the point was in the order polytope. Thus this point is contained in one of the simplices in the triangulation.

The standard triangulation of the order polytope results in a well known formula for its volume. Write $e(P)$ for the number of linear extension of a poset P. Then we have,

$$\mathrm{Vol}(\mathscr{O}(P)) = \frac{E(P)}{n!}.$$

We can interpret our standard triangulation in terms of the combinatorial description of the faces of the order polytope. A linear extension of the poset induces a sequence of $d+1$ filters in the poset. These are

$$\emptyset, \{i_1\}, \{i_1, i_2\}, \ldots, \{i_1, \ldots, i_n\}.$$

Visually this corresponds to drawing $d+1$ distinct lines through the extended poset which do not intersect.

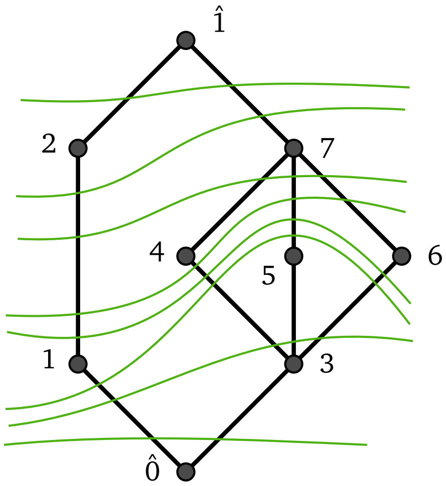

Figure 6.1: A linear extension of a poset represented as a sequence of filters. Here the total order is $3 \preceq 5 \preceq 1 \preceq 6 \preceq 4 \preceq 7 \preceq 2$.

The standard triangulation of the order polytope is unimodular. Furthermore it is a pulling triangulation. To see this, consider pulling from the origin. The facets which do not contain this vertex are exactly those corresponding to maximal elements in the poset. Choosing a facet to triangulate over then chooses a maximal element in the total order. This maximal element is added to the empty filter, creating the next filter is the nested sequence. The process then recurses by collapsing the chosen maximal cover relation until an entire simplex is obtained. Each choice of maximal cover relation to collapse then gives a different simplex in the triangulation.

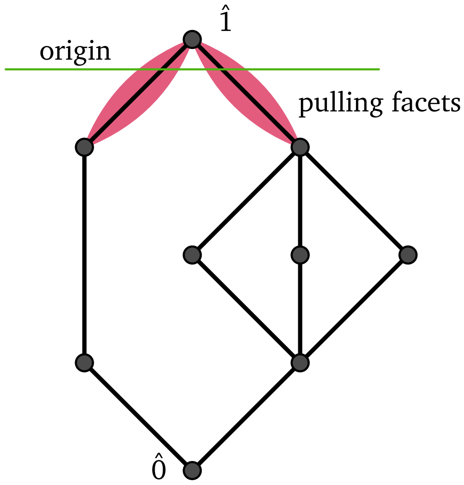

Figure 6.2: Beginning a pulling triangulation in the order polytope.

6.2 Chain Polytopes and the Transfer Map

Two polytopes which arise from posets are the order polytope and the chain polytope. In [12] Richard Stanley introduced these two polytopes and described a nice function between them. This so called transfer map is, amongst other things, continuous, piecewise-linear and volume preserving. Importantly it transfers the triangulation of the order polytope to a triangulation of the chain polytope.

Although we haven't seen the chain polytope before, it is actually a familiar object. The polytope has an $\mathscr{H}$-representation arising from chains in the polytope and a $\mathscr{V}$-representation coming from antichains.

Definition 6.1 (Chain Polytope).
Given a poset P, the chain polytope $\mathscr{C}(P)$ is defined by points $x \in \mathbb{R}^n$ with

$$0 \leq x_{i_1} + \cdots + x_{i_m} \leq 1$$

for all chains $i_1 \preceq \cdots \preceq i_m$ in the poset. Equivalently, the polytope is given by,

$$\mathscr{C}(P) = \left\{ e_{i_1} + \cdots + e_{i_m} : \{i_1, \ldots, i_m\} \text{ is an antichain in } P \right\}.$$

Actually, these chain polytopes are just examples of the independent set polytope. The comparability graph of a poset is a graph on the elements of the poset with two elements connected when they are comparable.

Definition 6.2 (Comparability Graph)**.**
Given a poset $(P, \preceq)$, the comparability graph is a graph $G = (V, E)$ with

$$V = P$$

and $\{x, y\} \in E$ when

$$x \preceq y \text{ or } y \preceq x.$$

Following this definition it is clear that chains in the poset are cliques in the comparability graph and antichains are independent sets. So the chain polytope is just the independent set polytope of the comparability graph. Note that comparability graphs are always perfect. This conforms with our earlier notions about when the independent set and clique polytopes are identical.

One question we asked is whether chain polytopes (and twisted order polytopes) are alcove polytopes. Alcove polytopes are any polytope which occurs between hyperplanes of a Coxeter complex transformed along a lattice. Equivalently (up to an affine transformation) polytopes where every normal vector is a root of an irreducible root system. Alcove polytopes are an already known class of two-level polytope. Neither chain polytopes not the twisted prisms of order polytopes are in general alcove polytopes. To see this consider posets formed by stacking three layers of elements with all elements in each layer covered by those in the above layer.

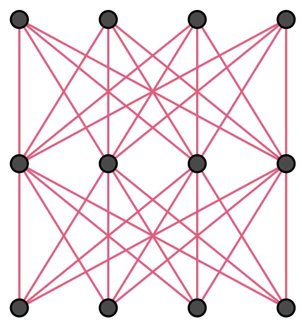

Figure 6.3: A poset formed by stacking three layers of elements.

If $n = 3m$ and the elements are distributed equally between the layers, then this poset has 3^m maximal chains. In dimension n the infinite families of irreducible root systems have roots

$$\pm e_i, \pm e_i \pm e_j$$

of which there are $2n + 2n(n-1)$. This grows quadratically in n and so for sufficiently high n is strictly less than the number of maximal chains in the given poset.

Transfer Map

Stanley defined a continuous piecewise-linear bijection between the order polytope and the chain polytope. This map does not preserve the face structure of the polytopes, however it does map the standard triangulation of the order polytope to a triangulation of the chain polytope.

> **Definition 6.3** (Transfer Map).
> Given a poset P, the transfer map $\phi : \mathcal{O}(P) \to \mathcal{C}(P)$ is defined by,
>
> $$\phi(x)_i = \min_{j \in P}\{x_i - x_j : j \preceq i\}.$$

For any simplex in the order polytope defined by $0 \leq x_{i_1} \leq \cdots \leq x_{i_n} \leq 1$, the transfer map is linear. The fact that this transfer map ϕ is a bijection follows from the existence of a well defined inverse ψ which is given by,

$$\psi(x)_i = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_m} : i_1 \preceq i_2 \preceq \cdots \preceq i_m = i \text{ is a chain in } P\}$$

One helpful way to interpret the transfer map is to consider its action on vertices of the two polytopes. Vertices of the order polytope are characteristic vectors of filters and vertices of the chain polytope are characteristic vectors of antichains. There is a well known bijection between filters and antichains which arises by mapping a filter to the set of its minimal elements. Stanley's transfer map realises this exact relation on the vertices of the two polytopes.

6.3 Extending the Triangulation to the Twisted Prism

We would like to extend the standard triangulation of the order polytope to a triangulation of its twisted prism. What exactly do we mean by extend? Firstly, restricting a triangulation of the twisted prism to the copies of $\mathcal{O}(P)$ and $-\mathcal{O}(P)$ induces a triangulation on each of them. We require that both of these induced triangulations are the standard triangulation of the order polytope. Secondly, our triangulation of the order polytope is a pulling triangulation and we would like the same to be true for the extended triangulation. If this is the case, then since the twisted order polytope is two-level, the triangulation is unimodular and hence we can determine the volume of the polytope from the number of simplices.

Maximal Simplices

We will describe the proposed triangulation by giving a combinatorial description of the maximal simplices. These can be computed purely by looking at the poset.

Consider any partition of the elements of the poset into two (possibly empty) sets. As a convenient terminology we will refer to these two sets as the red and green elements. For each of the colours, take a linear extension of the subposet induced by the elements of that colour. This induces two nested sequences of filters in P. For each colour this sequence arises by beginning with the empty filter and adding a maximal element of the same colour to the previous filter. To ensure that the result is a filter, we don't just add the element but its entire upset. This gives $n+2$ filters in total, two copies of the empty filter and then an additional filter for each of the n elements of the poset.

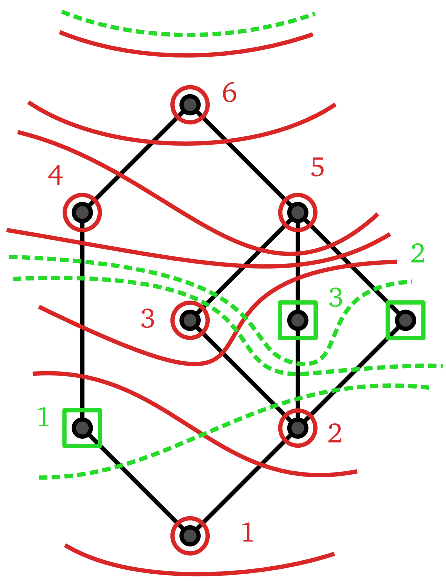

Figure 6.4: An example simplex in a twisted order polytope.

Designating the green filters as vertices in the copy of $\mathcal{O}(P)$ and red filters as vertices in the copy of $-\mathcal{O}(P)$ we obtain a simplex contained within the twisted order polytope. Note that this is a simplex as the $n+2$ points are affinely independent. To see this observe that, with the exception of the two copies of the empty filter, each of the filters is defined by some unique minimal element.

Triangulation of the Twisted Chain Polytope

Despite searching for a triangulation of the twisted order polytope, it is actually easier to prove that our maximal simplices form a triangulation of the twisted chain polytope. This is because the simplices, as described, make ready use of subposets. Although we can look at the order polytope induced by a subposet, there is no clear way to embed this within the original order polytope. In the chain polytope this is not a problem. Here vertices are characteristic vectors of antichains in the poset and every antichain in a subposet is also an antichain of the original poset.

Proposition 6.2 (Triangulation of the Twisted Chain Polytope)**.**
The set of maximal simplices described above form a triangulation of the twisted chain polytope.

The collection of partitions of the poset elements into two parts creates a subdivision of the twisted chain polytope. Here the regions in the subdivision are the convex hulls of the vertices corresponding to antichain in the two induced sub-posets. Green vertices are given height one and red vertices are given height zero. This corresponds to partitioning the polytope into regions depending on the signs of the coordinates.

Observe that any two of the regions in the proposed subdivision will not intersect in their interior. To see this, note that any two distinct partitions of the elements of the poset will have some element i which is red in one partition and green in the other. In the twisted prism, the partition where i is green will satisfy $x_i \geq 0$ and the partition where i is red will satisfy $x_i \leq 0$. Thus the two regions have disjoint interiors.

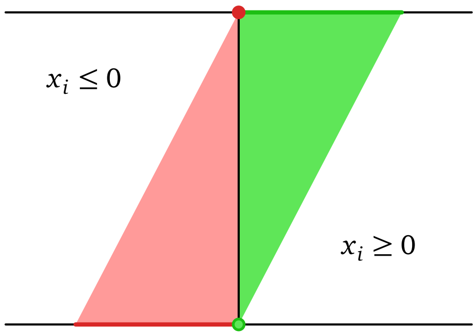

Figure 6.5: Distinct partitions give rise to disjoint regions.

Secondly, any two regions in the subdivision intersect in a common face. Consider partitions $P = A_1 \cup A_2$ and $P = B_1 \cup B_2$. The top and bottom levels of the two regions induced by these subdivisions will intersect in the chain subpolytopes $A_1 \cap B_1$ and $A_2 \cap B_2$ respectively. These will be facets of the top and bottom obtained by setting the elements in $A_1 \Delta B_1$ and $A_2 \Delta B_2$ respectively to zero. However,

$$A_1 \Delta B_1 = A_1^C \Delta B_1^C = A_2 \Delta B_2.$$

Thus the intersection of the two regions in the subdivision is a common face given by setting some coordinates to zero.

Finally the proposed subdivision covers the entire polytope. To see this, note that the intersection of the twisted chain polytope with a coordinate hyperplane $x_i = 0$ $(i = 1, \ldots, n)$ is itself a twisted chain polytope. The vertices of this intersection are a subset of the vertices of the ambient polytope as the subset of every antichain is an antichain. As such, any point in $\mathscr{T}(\mathscr{C}(P))|_{x_i \geq 0}$ can be written as a convex combination of vertices in $\mathscr{T}(\mathscr{C}(P))|_{x_i \geq 0}$. That is, written as a sum of vertices with non-negative ith coordinate. Applying this to all coordinates means that any point in a region defined by the signs of its coordinates can be written as a combination of vertices with the same signs.

The two components of each region in the subdivision, corresponding to the red and green parts, are orthogonal, since each is zero in all the coordinates the other is not. Furthermore they lie at different heights and are hence skew. As such the convex hull of the two polytopes is their join. Triangulations of joins are exactly the joins of maximal simplicies from a triangulation of each of the two parts. For a proof of this see the book on triangulations by Loera, Rambau and Santos [2]. We can triangulate each of the levels by taking the standard triangulation of the order polytopes and applying Stanley's map. The join then extends this to a triangulation of each region of the subdivision. Note that the simplices match on the boundaries of these regions as each intersection of regions is itself a join of chain polytopes.

This process results in exactly the maximal simplices described earlier.

6.4 Extending the Map to the Twisted Prisms

Stanley's map is a continuous piecewise-linear volume-preserving bijection between the order and chain polytopes. We would like to extend this function to a map between the two twisted prisms.

To define the map we begin with the triangulation of the twisted chain polytope. Each region in the subdivision of the twisted chain polytope is defined as the join of two orthogonal chain polytopes C_1 and C_2 corresponding to the signs of coordinates. Thus, any point x in the twisted chain polytope can be written as

$$x = x^+ + x^-$$

with

$$x^+ \in \frac{1}{\lambda} C_1$$

and

$$x^- \in \frac{-1}{1-\lambda} C_2$$

for appropriate chain polytopes C_1 and C_2. Applying Stanley's map individually to the positive and negative coordinates gives a point in the twisted order polytope.

$$\Psi : \mathscr{T}\mathscr{C}(P) \to \mathscr{T}\mathscr{O}(P)$$

$$\begin{aligned}\Psi(t,x) &= (t, \psi(x^+)) + (0, -\psi(-x^-)) \\ &= (t, \psi(x^+) - \psi(-x^-))\end{aligned}$$

This is a piecewise-linear map from the chain polytope to the order polytope. Indeed

$$\frac{1}{t}\psi(x^+) \in \mathcal{O}(P)$$

and

$$\frac{-1}{1-t}\psi(-x^-) \in -\mathcal{O}(P).$$

To demonstrate that this extended map is a bijection we need to construct an inverse. Recall the explicit definition of Stanley's map,

$$\psi(x)_i = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_m} : i_1 \preceq i_2 \preceq \cdots \preceq i_m = i \text{ is a chain in } P\}.$$

Note that each chain over which we take the maximum contains the element corresponding to the coordinate under focus. As such we can rewrite the map as

$$\psi(x)_i = x_i + \max\{x_{i_1} + \cdots + x_{i_m} : i_1 \prec \cdots \prec i_m \prec i \text{ is a chain in } P\}.$$

This gives the following equivalent formulation of the extended map,

$$\begin{aligned}\Psi(t,x)_i &= \left(t, x_i^+ + \max_{i_1 \prec \cdots \prec i_m \prec i}\left(x_{i_1}^+ + \cdots + x_{i_m}^+\right) + x_i^- + \min_{i_1 \prec \cdots \prec i_k \prec i}\left(x_{i_1}^- + \cdots + x_{i_k}^-\right)\right)\\ &= \left(t, x_i + \max_{i_1 \prec \cdots \prec i_m \prec i}\left(x_{i_1}^+ + \cdots + x_{i_m}^+\right) + \min_{i_1 \prec \cdots \prec i_k \prec i}\left(x_{i_1}^- + \cdots + x_{i_k}^-\right)\right).\end{aligned}$$

We can rearrange this to express a partial inverse.

$$x_i = \Psi(t,x)_i - \max_{i_1 \prec \cdots \prec i_m \prec i}\left(x_{i_1}^+ + \cdots + x_{i_m}^+\right) + \min_{i_1 \prec \cdots \prec i_k \prec i}\left(x_{i_1}^- + \cdots + x_{i_k}^-\right)$$

Importantly, to calculate the ith coordinate of the inverse function we only need to know the coordinates j in the downset of i. Furthermore, note that the map is the identity on coordinates corresponding to minimal elements of the poset. Together this allows us to recursively compute the entire inverse. As such the extended transfer map is a bijection.

Triangulation of Order Polytope

The set of maximal simplices described before also forms a triangulation of the twisted order polytope. Since the extended transfer map is continuous and piecewise linear, it maps the triangulation of the twisted chain polytope to a triangulation of the twisted order polytope. The extended map complies with the original connection between filters and antichains and as such the image triangulation is exactly the set of maximal simplices from before.

Corollary 6.1.
The maximal simplices described above form a triangulation of the twisted order polytope.

Volume Formulas

We can use the triangulation presented in the previous section to give an interesting formula for the volume of the twisted order (and chain) polytopes. Recall that we write $e(P)$ for the number of linear extensions of a poset. The number of simplices in the standard triangulation of the order polytope is exactly $e(P)$. Using the same notation, the number of simplices in the twisted order polytope is

$$\sum_{P_1 \uplus P_2 = P} e(P_1) \cdot e(P_2)$$

Given that we embed the polytopes as $0/1$ polytopes this then gives a formula for the volume of the twisted order and chain polytopes.

Corollary 6.2 (Volume of the Twisted Order Polytope)**.**

$$\mathrm{Vol}(\mathscr{T}(\mathscr{O}(P))) = \mathrm{Vol}(\mathscr{T}(\mathscr{C}(P))) = \frac{1}{(n+1)!} \sum_{P_1 \uplus P_2 = P} e(P_1) \cdot e(P_2)$$

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